Math 2250-004
Week 13.
Mon Apr 10: We will use last Friday's notes to discuss of finish discussing impulse forcing and the convolution table entry for Laplace transforms. If we have time, we'll move ahead into Tuesday's notes.

Because it is so important it's worth highlighting that for any non-homogeneous forced oscillation problem, the solution function $x(t)$ for the IVP that starts with zero initial conditions can be written as a convolution integral: For the IVP

$$a\,x'' + b\,x' + c\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Then in Laplace land, this equation is equivalent to

$$a\,s^2\,X(s) + b\,s\,X(s) + c\,X(s) = F(s)$$
$$\Rightarrow X(s)\left(a\,s^2 + b\,s + c\right) = F(s)$$
$$\Rightarrow X(s) = F(s)\cdot\frac{1}{a\,s^2 + b\,s + c} = F(s)\cdot W(s)$$

for

$$W(s) := \frac{1}{a\,s^2 + b\,s + c}.$$

The inverse Laplace transform $w(t) = \mathcal{L}^{-1}\{W(s))\}(t)$ is called the "weight function" of the given differential equation. Notice (check!) that $w(t)$ is the solution to the homogeneous DE IVP

$$a\,x'' + b\,x' + c\,x = 0$$
$$x(0) = 0$$
$$x'(0) = 1$$

Because of the convolution table entry

$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

the solution (for ANY forcing function $f(t)$) is given by

$$x(t) = \int_0^t f(\tau)\,w(t-\tau)\,d\tau.$$

(With non-zero initial conditions there would be homogeneous solution terms as well. In the case of damping these terms would be transient.) Notice that this says that $x(t)$ depends on the values of the forcing function $f(\tau)$ for the previous times $0 \le \tau \le t$, weighted by $w(t-\tau)$, $t \ge t-\tau \ge 0$. That the non-homogenous solutions can be constructed from the homogeneous ones via this convolution is a special case of "Duhamel's Principle", which applies to linear DE's and linear PDE's:

https://en.wikipedia.org/wiki/Duhamel%27s_principle

Math 2250-004
Tues Apr 11

<u>6.1-6.2</u> Eigenvalues and eigenvectors for square matrices.

The study of eigenvalues and eigenvectors is a return to matrix linear algebra, and the concepts we discuss will help us study linear <u>systems of differential equations</u>, in Chapter 7. Such systems of DE's arise naturally in the contexts of

- coupled input-output models, with several components.
- coupled mass-spring or RLC circuit loops, with several components.

To introduce the idea of eigenvalues and eigenvectors we'll first think geometrically.

<u>Example</u> Consider the matrix transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

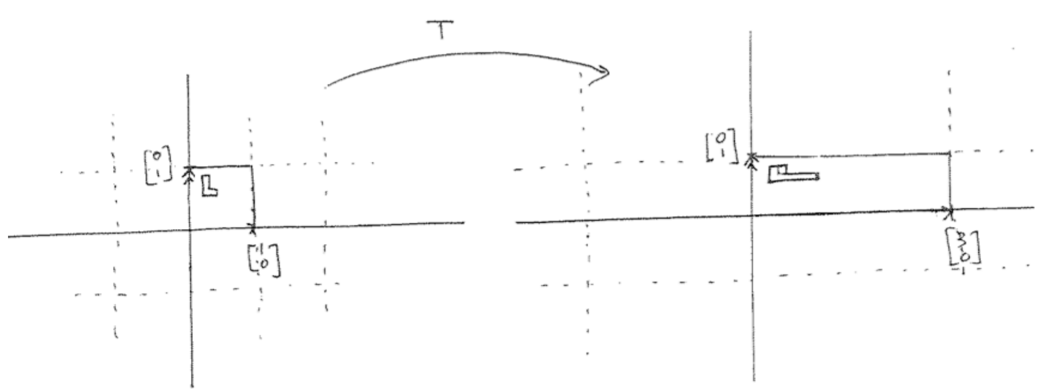
Notice that for the standard basis vectors $\underline{\boldsymbol{e}}_1 = [1, 0]^T, \underline{\boldsymbol{e}}_2 = [0, 1]^T$

$$T(\underline{\boldsymbol{e}}_1) = 3\underline{\boldsymbol{e}}_1$$
$$T(\underline{\boldsymbol{e}}_2) = \underline{\boldsymbol{e}}_2 \ .$$

The facts that T is linear and that it transforms $\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2$ by scalar multiplying them, lets us understand the geometry of this transformation completely:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(x_1\underline{\boldsymbol{e}}_1 + x_2\underline{\boldsymbol{e}}_2) = x_1T(\underline{\boldsymbol{e}}_1) + x_2T(\underline{\boldsymbol{e}}_2)$$
$$= x_1(3\underline{\boldsymbol{e}}_1) + x_2(1\underline{\boldsymbol{e}}_2) \ .$$

In other words, T stretches by a factor of 3 in the $\underline{\boldsymbol{e}}_1$ direction, and by a factor of 1 in the $\underline{\boldsymbol{e}}_2$ direction, transforming a square grid in the domain into a parallel rectangular grid in the image:



Exercise 1) Do a similar geometric analysis and sketch for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 2) And for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Definition: If $A_{n \times n}$ and if $A\,\underline{v} = \lambda\,\underline{\boldsymbol{v}}$ for a scalar λ and a vector $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ then $\underline{\boldsymbol{v}}$ is called an eigenvector of A, and λ is called the eigenvalue of $\underline{\boldsymbol{v}}$. (In some texts the words characteristic vector and characteristic value are used as synonyms for these words.)

• In the three examples above, the standard basis vectors (or multiples of them) were eigenvectors, and the corresponding eigenvalues were the diagonal matrix entries. A non-diagonal matrix may still have eigenvectors and eigenvalues, and this geometric information can still be important to find. But how do you find eigenvectors and eigenvalues for non-diagonal matrices? ...

Exercise 3) Try to find eigenvectors and eigenvalues for the non-diagonal matrix, by just trying random input vectors $\underline{\boldsymbol{x}}$ and computing $A\,\underline{\boldsymbol{x}}$.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

How to find eigenvalues and eigenvectors (including eigenspaces) systematically:

If

$$A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$$
$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \boldsymbol{\lambda}\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$
$$\Leftrightarrow A\,\underline{\boldsymbol{v}} - \lambda\,I\,\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

where I is the identity matrix.

$$\Leftrightarrow (A - \lambda\,I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}.$$

As we know, this last equation can have non-zero solutions $\underline{\boldsymbol{v}}$ if and only if the matrix $(A - \lambda\,I)$ is not invertible, i.e.

$$\Leftrightarrow det(A - \lambda\,I) = 0.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in λ

$$p(\lambda) = det(A - \lambda\,I).$$

If $A_{n \times n}$ then $p(\lambda)$ will be degree n. This polynomial is called the characteristic polynomial of the matrix A.

- λ_j can be an eigenvalue for some non-zero eigenvector $\underline{\boldsymbol{v}}$ if and only if it's a root of the characteristic polynomial, i.e. $p(\lambda_j) = 0$. For each such root, the homogeneous solution space of vectors $\underline{\boldsymbol{v}}$ solving

$$(A - \lambda_j\,I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

will be eigenvectors with eigenvalue λ_j. This subspace of eigenvectors will be at least one dimensional, since $(A - \lambda_j\,I)$ does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

Notation: The subspace of eigenvectors for eigenvalue λ_j is called the λ_j eigenspace, and denoted by E_{λ_j}. (We include the zero vector in E_{λ_j}.) The basis of eigenvectors is called an eigenbasis for E_{λ_j}.

Exercise 4) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 3.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

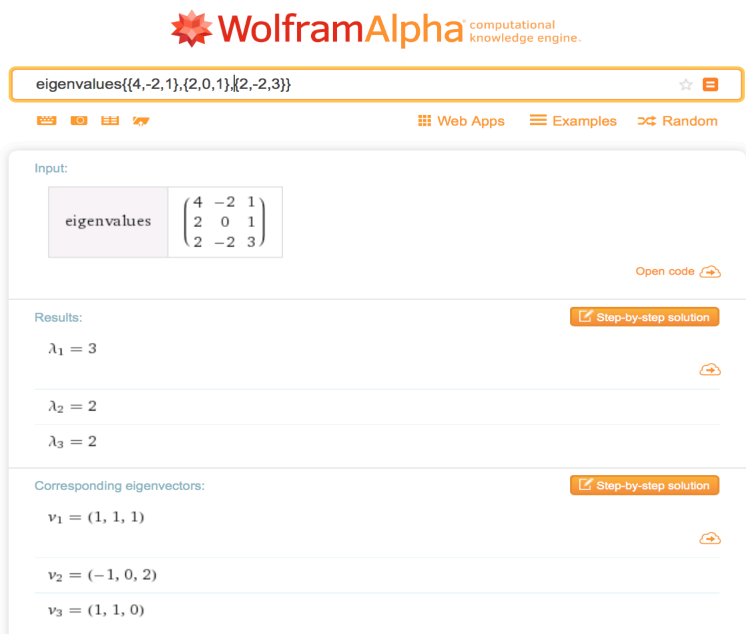
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 5) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues.
(ii) for each eigenvalue, find bases for the corresponding eigenspaces.
(iii) Can you describe the transformation $T(\underline{\mathbf{x}}) = B\underline{\mathbf{x}}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?

Your solution will be related to the output below:



In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$$

almost as well as if A is a diagonal matrix. This is actually something that does not always happen for a matrix A. When it does happen, we say that A is diagonalizable. Here's an example of a matrix which is NOT diagonalizable:

Exercise 7: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Math 2250-004
Wed Apr 12
6.1-6.2 Eigenvalues and eigenvectors for square matrices; diagonalizability of matrices.

Recall from yesterday,

Definition: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for some scalar λ and vector $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ then $\underline{\boldsymbol{v}}$ is called an eigenvector of A, and λ is called the eigenvalue of $\underline{\boldsymbol{v}}$ (and an eigenvalue of A).

- For general matrices, the eigenvector equation $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ can be rewritten as

$$(A - \lambda I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \ .$$

The only way such an equation can hold for $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ is if the matrix $(A - \lambda I)$ does not reduce to the identity matrix. In other words - $det(A - \lambda I)$ must equal zero. Thus the only possible eigenvalues associated to a given matrix must be roots λ_j of the characteristic polynomial

$$p(\lambda) = det(A - \lambda I) \ .$$

- So, the first step in finding eigenvectors for A is actually to find the eigenvalues - by finding the characteristic polynomial and its roots λ_j .

- For each root λ_j the matrix $A - \lambda_j I$ will not reduce to the identity, and the solution space to

$$\left(A - \lambda_j I\right)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

will be at least one-dimensional, and have a basis of one or more eigenvectors. Find such a basis for this λ_j eigenspace $E_{\lambda = \lambda_j}$ by reducing the homogeneous matrix equation

$$\left(A - \lambda_j I\right)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \ ,$$

backsolving, and extracting a basis. We can often "see" and eigenvector by realizing that homogeneous solutions to a matrix equation correspond to column dependencies.

- Finish any leftover exercises from Tuesday

Exercise 1) If your matrix A is diagonal, the general algorithm for finding eigenspace bases just reproduces the entries along the diagonal as eigenvalues, and the corresponding standard basis vectors as eigenspace bases. (Recall our diagonal matrix examples from yesterday, where the standard basis vectors were eigenvectors. This is typical for diagonal matrices.) Illustrate how this works for a 3×3 diagonal matrix, so that in the future you can just read of the eigendata if the matrix you're given is (already) diagonal:

$$A := \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

step 1) Find the roots of the characteristic polynomial $det(A - \lambda I)$.
step 2) Find the eigenspace bases, assuming the values of a_{11}, a_{22}, a_{33} are distinct (all different). What if $a_{11} = a_{22}$ but these values do not equal a_{33} ?

In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\mathbf{x}}) = A\,\underline{\mathbf{x}}$$

almost as well as if A is a diagonal matrix, and so we call such matrices diagonalizable. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word diagonalizable to describe such matrices.

Use the $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 1, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column:

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

In other words,

$$B\,P = P\,\mathrm{D}\,,$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P). Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P\,\mathrm{D}\,P^{-1} \text{ and } P^{-1}BP = \mathrm{D}.$$

Exercise 2) Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 | \lambda_2\underline{\boldsymbol{v}}_2 | \cdots | \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ : & : & \cdots & : \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, not all matrices are diagonalizable:

Exercise 3) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable.

Facts about diagonalizability (see text section 6.2 for complete discussion, with reasoning):

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \dots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \dots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \leq dim\left(E_{\lambda = \lambda_j}\right) \leq k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called defective.
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

- In the special case that A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ each eigenspace is forced to be

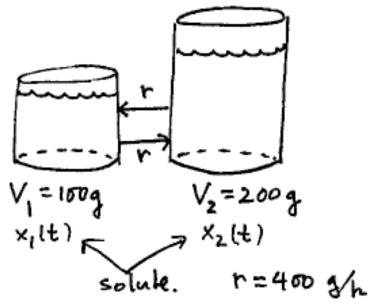
1-dimensional since $k_1 + k_2 + \dots + k_n = n$ so each $k_j = 1$. Thus A is automatically diagonalizable as a special case of the second bullet point.

Exercise 4) How do the examples from today and yesterday compare with the general facts about diagonalizability?

Math 2250-004
Fri Apr 14

7.1 Systems of differential equations - to model multi-component systems via compartmental analysis:
http://en.wikipedia.org/wiki/Multi-compartment_model

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t)$, $x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1$, $x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

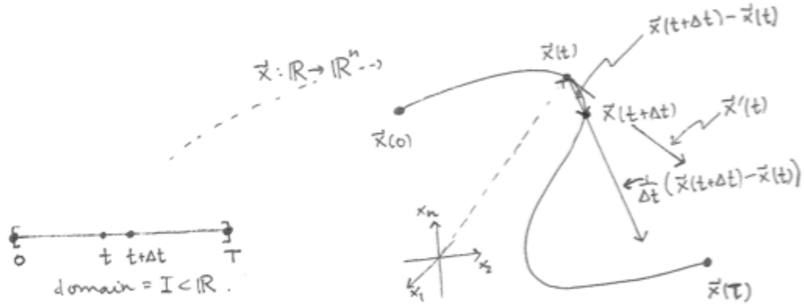
The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

- We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\boldsymbol{x}}(t)$:

- From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\boldsymbol{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\boldsymbol{x}}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\underline{\boldsymbol{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! You just convert the scalar quantities in the code into vector quantities. And this is what numerical solvers do.)

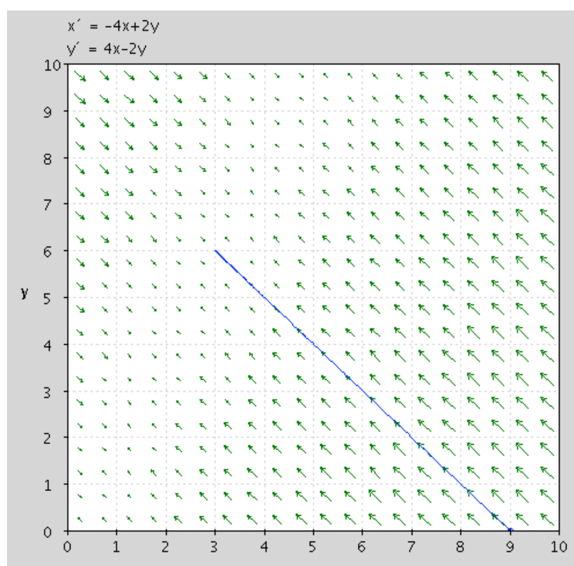
Exercise 2) Return to the page 1 tank example

$$x_1{}'(t) = -4\,x_1 + 2\,x_2$$
$$x_2{}'(t) = 4\,x_1 - 2\,x_2$$
$$x_1(0) = 9$$
$$x_2(0) = 0$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time *t*. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?



First order systems of differential equations of the form

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as

$$\underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}} = \underline{\boldsymbol{f}}(t)$$

or

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}} + \underline{\boldsymbol{f}}(t)$$

Notice that the operator on vector-valued functions $\underline{\mathbf{x}}(t)$ defined by

$$L(\underline{\mathbf{x}}(t)) := \underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}}(t)$$

is linear, i.e.

$$L(\underline{\mathbf{x}}(t) + \underline{\mathbf{y}}(t)) = L(\underline{\mathbf{x}}(t)) + L(\underline{\mathbf{y}}(t))$$
$$L(c\,\underline{\mathbf{x}}(t)) = c\,L(\underline{\mathbf{x}}(t)).$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\underline{\mathbf{x}}'(t) - A\,\underline{\mathbf{x}} = \underline{\boldsymbol{f}}(t)$$

will be of the form

$$\underline{\mathbf{x}} = \underline{\mathbf{x}}_P + \underline{\mathbf{x}}_H$$

where $\underline{\mathbf{x}}_P$ is any single particular solution and $\underline{\mathbf{x}}_H$ is the general homogeneous solution.

Exercise 3) In the case that A is a constant matrix (i.e. entries don't depend on t), consider the homogeneous problem

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}.$$

Look for solutions of the form

$$\underline{\mathbf{x}}(t) = e^{\lambda t}\underline{\mathbf{v}},$$

where $\underline{\mathbf{v}}$ is a constant vector. Show that $\underline{\mathbf{x}}(t) = e^{\lambda t}\underline{\mathbf{v}}$ solves the homogeneous DE system if and only if v is an eigenvector of A, with eigenvalue λ, i.e. $A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}$.

Hint: In order for such an $\underline{\mathbf{x}}(t)$ to solve the DE it must be true that

$$\underline{\mathbf{x}}'(t) = \lambda\, e^{\lambda t}\underline{\mathbf{v}}$$

and

$$A\,\underline{\mathbf{x}}(t) = A\, e^{\lambda t}\underline{\mathbf{v}} = e^{\lambda t} A\,\underline{\mathbf{v}}$$

Set these two expressions equal.

Exercise 4) Use the idea of Exercise 3 to solve the initial value problem of Exercise 2!! Compare your solution $\underline{\mathbf{x}}(t)$ to the parametric curve drawn by pplane, that we looked at a couple of pages back.

Exercise 5) Lessons learned from tank example: What condition on the matrix $A_{n \times n}$ will allow you to uniquely solve every initial value problem

$$\underline{\mathbf{x}}'(t) = A\,\underline{\mathbf{x}}$$
$$\underline{\mathbf{x}}(0) = \underline{\mathbf{x}}_0 \in \mathbb{R}^n$$

using the method in Exercise 3-4 ? Hint: Chapter 6. (If that condition fails there are other ways to find the unique solutions.)