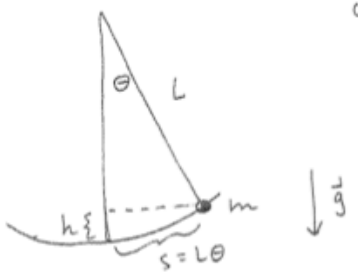
Math 2250-004 Week 11 March 27-31

Mon Mar 27: Continue and maybe finish Friday's notes on section 5.6, forced oscillation phenomena

Tues Mar 28: Pendulum and mass-spring experiments from last Tuesday (notes included here, though). Then begin Laplace Transform, 10.1-10.2

<u>Experiment discussion</u>: Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \frac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



$\theta = \theta(t)$

conservative system $KE + PE = \text{const.}$

$\frac{1}{2} m v^2 + m g h = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2} m L^2 (\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$\underbrace{mL\theta'}_{\neq 0 \text{ except at isolated times}}(L\theta'' + g\sin\theta) \equiv 0$

$\rightsquigarrow$ deduce eqtn of motion is $\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

linearize

$\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

non-linear DE

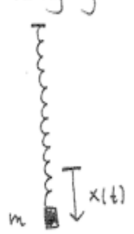
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin\theta \approx \theta$ θ small

is excellent approx

(alternating series test)

2) hanging mass-spring:



$m x'' = -kx$

$m x'' + kx = 0$

$\boxed{x'' + \frac{k}{m}x = 0}$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
> restart :
  Digits := 4 :
```

> $L := 1.53;$
> $g := 9.806;$
> $\omega := \sqrt{\frac{g}{L}}$; # *radians per second*
> $f := evalf(\omega/(2\cdot \mathrm{Pi}))$; # *cycles per second*
> $T := 1/f$; # *seconds per cycle*

$$L := 1.53$$
$$g := 9.806$$
$$\omega := 2.531629974$$
$$f := 0.4029214244$$
$$T := 2.481873486 \quad \textbf{(1)}$$

Experiment:

Mass-spring:
compute Hooke's constant:

> $98.7 - 83.4$; *#displacement from extra 50g*

$$15.3 \quad \textbf{(2)}$$

> $k := \frac{.05\cdot 9.806}{.153}$; # *solve k·x=m·g for k.*

$$k := 3.204575163 \quad \textbf{(3)}$$

> $m := .1$; # *mass for experiment is 100g*
> $\omega := \sqrt{\frac{k}{m}}$; # *predicted angular frequency*
> $f := evalf\left(\frac{\omega}{2\cdot \mathrm{Pi}}\right)$; # *predicted frequency*
> $T := \frac{1}{f}$; # *predicted period*

$$m := 0.1$$
$$\omega := 5.660896716$$
$$f := 0.9009596945$$
$$T := 1.109927565 \quad \textbf{(4)}$$

Experiment:

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>
Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}\,.$$

$$PE_{work} = \int_0^x k\,s\,\mathrm{d}s = \frac{1}{2}k\,x^2, \quad KE_{mass} = \frac{1}{2}m\,(x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\,x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ, as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2}\left(\mu\,x'(t)\right)^2\left(m_{spring}\,\mathrm{d}\mu\right)$$

$$= \frac{1}{2}m_{spring}(x'(t))^2\int_0^1 \mu^2 d\,\mu = \frac{1}{6}m_{spring}(x'(t))^2\ .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3}m_{spring}\right)(x'(t))^2 + \frac{1}{2}k\,x^2 = \frac{1}{2}M(x'(t))^2 + \frac{1}{2}k\,x^2\,,$$

where

$$M = m + \frac{1}{3}m_{spring}$$

$\mathrm{D}_t(TE) = 0 \Rightarrow$

$$M\,x'(t)\,x''(t) + k\,x(t)\,x'(t) = 0\,.$$
$$x'(t)\,(M\,x'' + k\,x) = 0\,.$$

Since $x'(t) = 0$ only at isolated t-values, we deduce that the corrected equation of motion is

$$(M\,x'' + k\,x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}}\ .$$

Does this lead to a better comparison between model and experiment?

```
> ms := .011;  # spring has mass 11g
  M := m + 1/3·ms;  # "effective mass"
```

$$ms := 0.011$$
$$M := 0.1036666667 \quad \textbf{(5)}$$

> $\omega := \sqrt{\frac{k}{M}}$; *# predicted angular frequency*
$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right)$; *# predicted frequency*
$T := \frac{1}{f}$; *# predicted period*

$$\omega := 5.559883146$$
$$f := 0.8848828855$$
$$T := 1.130093051 \tag{6}$$

>

10.1-10.2 Laplace transform, and application to DE IVPs, including Chapter 5.

- This new material having to do with the Laplace Transform will help you review and solidify the ideas in Chapter 5. It will not be tested on the second midterm, but will show up on the final exam.

- The Laplace transform is a linear transformation "$\mathcal{L}$" that converts piecewise continuous functions $f(t)$, defined for $t \ge 0$ and with at most exponential growth ($|f(t)| \le Ce^{Mt}$ for some values of C and M), into functions $F(s)$ defined by the transformation formula

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\, dt\,.$$

- Notice that the integral formula for $F(s)$ is only defined for sufficiently large s, and certainly for $s > M$, because as soon as $s > M$ the integrand is decaying exponentially, so the improper integral from $t = 0$ to ∞ converges.
- The convention is to use lower case letters for the input functions and (the same) capital letters for their Laplace transforms, as we did for $f(t)$ and $F(s)$ above. Thus if we called the input function $x(t)$ then we would denote the Laplace transform by $X(s)$.

Taking Laplace transforms seems like a strange thing to do. And yet, the Laplace transform $\mathcal{L}$ is just one example of a collection of useful "integral transforms". $\mathcal{L}$ is especially good for solving IVPs for linear DEs, as we shall see starting today. Other famous transforms - e.g. Fourier series and Fourier transform are extremely important in studying linear partial differential equations, as you will see in e.g. Math 3140, 3150, physics, engineering, or Wikipedia.

Exercise 1) Use the definition of Laplace transform

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\, dt$$

to check the following facts, which you will also find inside the front cover of your text book.

a) $\mathcal{L}\{1\}(s) = \dfrac{1}{s} \quad (s > 0)$

b) $\mathcal{L}\{e^{\alpha t}\}(s) = \dfrac{1}{s-\alpha} \quad (s > \alpha \text{ if } \alpha \in \mathbb{R},\ s > a \text{ if } \alpha = a + k\,i \in \mathbb{C})$

c) Laplace transform is linear, i.e.

$$\mathcal{L}\{f_1(t) + f_2(t)\}(s) = F_1(s) + F_2(s)\,.$$
$$\mathcal{L}\{c f(t)\}(s) = c\,F(s).$$

d) Use linearity and your work above to compute $\mathcal{L}\{3 - 4\,e^{-2t}\}(s)$.

For the DE's like we've just been studying in Chapter 5, the following Laplace transforms are very important:

Exercise 2) Use complex number algebra, including Euler's formula, linearity, and the result from 1b that

$$\mathcal{L}\{e^{(a+k i)t}\}(s) = \frac{1}{s-(a+k i)}$$

to verify that

a) $\mathcal{L}\{\cos(k t)\}(s) = \dfrac{s}{s^2+k^2}$

b) $\mathcal{L}\{\sin(k t)\}(s) = \dfrac{k}{s^2+k^2}$

c) $\mathcal{L}\left\{e^{a t}\cos\left(k t\right)\right\}(s) = \dfrac{s-a}{(s-a)^2+k^2}$

d) $\mathcal{L}\{e^{a t}\sin(k t)\}(s) = \dfrac{k}{(s-a)^2+k^2}$.

(Notice that if we tried doing these Laplace transforms directly from the definition, the integrals would be messy but we could attack them via integration by parts or integral tables.)

It's a theorem (hard to prove but true) that a given Laplace transform $F(s)$ can arise from at most one piecewise continuous function $f(t)$. (Well, except that the values of f at the points of discontinuity can be arbitrary, as they don't affect the integral used to compute $F(s)$.) Therefore you can read Laplace transform tables in either direction, i.e. not only to deduce Laplace transforms, but inverse Laplace transforms $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$ as well.

Exercise 3) Use the Laplace transforms we've computed and linearity to compute

$$\mathcal{L}^{-1}\left\{\frac{7}{s} + \frac{1}{s^2+16} - \frac{10\,s}{s^2+16}\right\}(t)\,.$$

$f(t)$ $\|f(t)\| \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
1	$\frac{1}{s}$ $(s > 0)$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$
$\cos(k\,t)$	$\frac{s}{s^2+k^2}$ $(s > 0)$
$\sin(k\,t)$	$\frac{k}{s^2+k^2}$ $(s > 0)$
$e^{a t}\cos(k\,t)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$
$e^{a t}\sin(k\,t)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$
$f'(t)$	$s\,F(s) - f(0)$
$f''(t)$	$s^2 F(s) - s\,f(0) - f'(0)$

Laplace transform table

The integral transforms of DE's and PDE's were designed to have the property that they convert the corresponding linear DE and PDE problems into algebra problems. For the Laplace transform it's because of these facts:

Exercise 4a) Use integration by parts and the definition of Laplace transform to show that

$$\mathcal{L}\{f'(t)\}(s) = s\,\mathcal{L}\{f(t)\}(s) - f(0) = s\,F(s) - f(0)\,.$$

4b) Use the result of a, applied to the function $f'(t)$ to show that

$$\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s\,f(0) - f'(0)\,.$$

4c) What would you guess is the Laplace transform of $f'''(t)$? Could you check this?

Here's an example of using Laplace transforms to solve DE IVPs, in the context of Chapter 5 and the mechanical (and electrical) application problems we just considered there.

Exercise 5) Consider the undamped forced oscillation IVP

$$x''(t) + 4\,x(t) = 10\cos(3\,t)$$
$$x(0) = 2$$
$$x'(0) = 1$$

If $x(t)$ is the solution, then both sides of the DE are equal. Thus the Laplace transforms are equal as well... .so, equate the Laplace transforms of each side and use algebra to find $\mathcal{L}\{x(t)\}(s) = X(s)$. Notice you've computed $X(s)$ without actually knowing $x(t)$! If you were happy to stay in "Laplace land" you'd be done. In any case, at this point you can use our table entries to find $x(t) = \mathcal{L}^{-1}\{x(t)\}(s)$.

(Notice that if your algebra skills are good you've avoided having to use the Chapter 5 algorithm of (i) find x_H (ii) find an x_P (iii) $x = x_P + x_H$ (iv) solve IVP.) Magic!

> $with(DEtools):$

> $dsolve(\{x''(t) + 4 \cdot x(t) = 10 \cdot \cos(3 \cdot t), x(0) = 2, x'(0) = 1\});$ # *to check*

$$x(t) = \frac{1}{2}\sin(2\,t) + 4\cos(2\,t) - 2\cos(3\,t) \quad \textbf{(7)}$$

<u>Exercise 6)</u> Use Laplace transform as above, to solve the IVP for the following underdamped, unforced oscillator DE:

$$x''(t) + 6\,x'(t) + 34\,x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

> $dsolve(\{x''(t) + 6 \cdot x'(t) + 34 \cdot x(t) = 0, x(0) = 3, x'(0) = 1\});$ # *to check*

$$x(t) = 2\,\mathrm{e}^{-3\,t}\sin(5\,t) + 3\,\mathrm{e}^{-3\,t}\cos(5\,t) \quad \textbf{(8)}$$

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Wed Mar 29
Finish Tuesday notes on Laplace transform introduction, and then review for Friday midterm exam.

Exam 2 Review Questions
Math 2250-004 March 2017

Our exam covers chapters 3.6, 4.1-4.4, 5.1-5.6 of the text. Only scientific calculators will be allowed on the exam.

• remember to get to class a few minutes early. The exam will be from 10:40 - 11:40 a.m. Graphing calculators are not allowed, only scientific ones. Symbolic answers are allowed, although a scientific calculator could give you confidence on an amplitude-phase calculation, for example.

I try to find problems that touch on most of these key topics. I've put $*$'s next to topics which have higher probabilities of appearing on my exams, although anything we've learned is fair game.

Chapter 3.6: Determinants. Approximately 10% of the exam could be related to this material.

Be able to compute $|A|$ for a square matrix A using cofactor expansions, row operations, or some combination of those procedures.

* What does the value of $|A|$ have to do with whether A^{-1} exists?

What's the magic formula for the inverse of a matrix? Can you work with this formula in the two by two or three by three cases? Can you use it?

Chapter 4.1-4.4

Approximately 40% of the exam will deal directly with this material....but much of Chapter 5 uses these concepts, so much more than 40% of the exam will be related to chapter 4. (And, as far as matrix and determinant computations go, you should remember everything you learned in Chapter 3.)

∗ **Do you know the key definitions?**

vector space

linear combination of a collection $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$ of vectors

linearly independent vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$

linearly dependent vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k$

span of a collection of vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k\}$

subspace of a vector space

basis of a vector space

dimension of a vector space

* **Subspace examples from Chapter 4**, involving the concepts above

  - the space of solutions $\underline{\boldsymbol{x}}$ to matrix equations $[A]\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$

  - $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k\}$

* **Subspace examples from Chapter 5**
  - solution space to homogeneous linear differential equation for e.g. $y = y(x)$ on an interval I, i.e. solutions to

$$L(y) := y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0$$

  - $span\{y_1, y_2, \ldots y_n\}$

* **What does it mean for a transformation $L : V \rightarrow W$ between vector spaces to be <u>linear</u>?**

Chapter 4 examples?

Chapter 5 examples?

* **What is the general solution to $L(y) = f$, if L is a linear transformation (or "operator"), in terms of particular and homogeneous solutions? Can you explain why?**

Chapter 5

About 50% of the exam will be related to this material.

* What is the **natural initial value problem** for n^{th} - order linear differential equation, i.e. the one that has unique solutions?

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = f(x)$$

$$y(x_0) = b_0$$

$$y'(x_0) = b_1$$

$$\vdots$$

$$y^{(n-1)}(x_0) = b_{n-1}$$

* What is the **dimension of the solution space to the homogeneous DE**

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 \;\; ?$$

Why?

* How can you tell if $\{y_1(x), y_2(x), \ldots y_n(x)\}$ is a **basis** for space of solutions to the homogeneous DE above?

How is your answer above related to a **Wronskian matrix** and the **Wronskian determinant**?

* How do you find the **general solution** to the **homogeneous constant coefficient linear DE**

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0\ ?$$

(your answer should involve the characteristic polynomial, Euler's formula, repeated roots, complex roots.) (Do you remember Euler's formula? Can you use it in the various ways we've seen? Do you remember the Taylor-Maclaurin series formula in general? For e^x, $\cos(x)$, $\sin(x)$ in particular?)

Do you know how to find **particular solutions to constant coefficient non-homogeneous linear DEs**

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f?$$

* using undetermined coefficients

* Can you solve IVP's for solutions $y(x)$ to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x)$$

$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

5.4, 5.6, EP3.7 Mechanical vibrations and forced oscillations; electrical circuit analog

$*$ What is the governing second order DE's for a **damped mass-spring configuration** (via Newton's second law)? units?

$*$ What case of the governing DE leads to **unforced damped oscillations?** What four phenomena have we discussed for unforced oscillation problems, and how to they arise? (Hint: one is undamped; three are damped.)

$*$ What are the forms of the solutions in these four cases?

$*$ Can you convert a linear combination $A\cos(\omega t) + B\sin(\omega t)$ in amplitude-phase form? Do you remember the addition angle formulas? Can you explain the physical properties of the solution?

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

* What are the possible phenomena with **forced undamped oscillations** (assuming the forcing function is sinusoidal) and how do they arise?

$$m\,x''(t) + k\,x(t) = F_0 \cos(\omega\, t) \text{ or } F_0 \sin(\omega\, t)$$

* What are the possible phenomena with **forced damped oscillations** (assuming the forcing function is sinusoidal)?

$$m\,x''(t) + c\,x'(t) + k\,x(t) = F_0 \cos(\omega\, t)$$

* Can you solve all initial value problems that arise in the situations above?

For deeper review, consult class notes and examples, quizzes, and homework+lab problems.