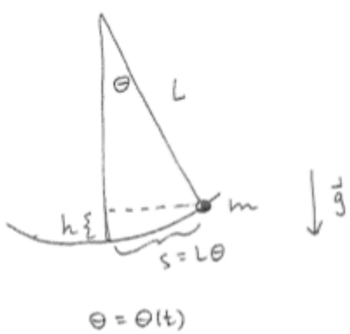
Math 2250-004
Week 10, March 20-24: 5.4-5.6

Mon Mar 20 Work through Fri Mar 10 notes on section 5.4: unforced mass-spring systems.

Tues Mar 21 If I'm able to obtain them from Physics, we'll do pendulum and mass-spring experiments :-). Then begin section 5.5 on finding particular solutions to inhomogeneous linear DE's. It's possible I won't have the experiments on Tuesday and that we'll move directly into section 5.5

Experiment discussion: Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \dfrac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



$\theta = \theta(t)$

conservative system KE + PE = const.

$\frac{1}{2} m v^2 + m g h = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t) \qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2} m L^2 (\theta'(t))^2 + m g L(1 - \cos(\theta(t))) \equiv \text{const}$

D_t: $m L^2 \theta'\theta'' + m g L (\sin\theta)\theta' \equiv 0$

$\underbrace{m L \theta'}_{\neq 0 \text{ except at isolated times}} (L\theta'' + g\sin\theta) \equiv 0$

⇝ deduce eqtn of motion is $\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

linearize:

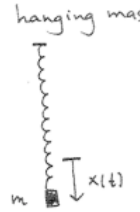
$\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

↘ non-linear DE
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \dots$
$\sin\theta \approx \theta$ θ small
is excellent approx
(alternating series test)

2) hanging mass-spring:



$m x'' = -k x$

$m x'' + k x = 0$

$\boxed{x'' + \frac{k}{m} x = 0}$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
> restart :
  Digits := 4 :
```

```
> L := 1.53;
  g := 9.806;
```

$\omega := \sqrt{\frac{g}{L}}$; *# radians per second*

$f :=$ *evalf*$(\omega / (2 \cdot \text{Pi}))$; *# cycles per second*

$T := 1/f$; *# seconds per cycle*

$$L := 1.53$$
$$g := 9.806$$
$$\omega := 2.531629974$$
$$f := 0.4029214244$$
$$T := 2.481873486 \qquad \textbf{(1)}$$

Experiment:

Mass-spring:

compute Hooke's constant:

$> 98.7 - 83.4;$ *#displacement from extra 50g*

$$15.3 \qquad \textbf{(2)}$$

$> k := \frac{.05 \cdot 9.806}{.153}$; *# solve k·x=m·g for k.*

$$k := 3.204575163 \qquad \textbf{(3)}$$

$> m := .1;$ *# mass for experiment is 100g*

$\omega := \sqrt{\frac{k}{m}}$; *# predicted angular frequency*

$f :=$ *evalf*$\left(\frac{\omega}{2 \cdot \text{Pi}}\right)$; *# predicted frequency*

$T := \frac{1}{f}$; *# predicted period*

$$m := 0.1$$
$$\omega := 5.660896716$$
$$f := 0.9009596945$$
$$T := 1.109927565 \qquad \textbf{(4)}$$

Experiment:

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

Improved mass-spring model

Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work} .$$

$$PE_{work} = \int_0^x k\, s\, \mathrm{d}s = \frac{1}{2} k x^2, \quad KE_{mass} = \frac{1}{2} m\, (x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\, x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2} (\mu\, x'(t))^2 \left(m_{spring}\, \mathrm{d}\mu\right)$$

$$= \frac{1}{2} m_{spring} (x'(t))^2 \int_0^1 \mu^2 d\,\mu = \frac{1}{6} m_{spring} (x'(t))^2 \ .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3} m_{spring}\right)(x'(t))^2 + \frac{1}{2} k x^2 = \frac{1}{2} M (x'(t))^2 + \frac{1}{2} k x^2 ,$$

where

$$M = m + \frac{1}{3} m_{spring}$$

$D_t(TE) = 0 \Rightarrow$

$$M\, x'(t)\, x''(t) + k\, x(t)\, x'(t) = 0 .$$
$$x'(t)\,(M\, x'' + k\, x) = 0 .$$

Since $x'(t) = 0$ only at isolated t-values, we deduce that the corrected equation of motion is

$$(M\, x'' + k\, x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3} m_{spring}}} \ .$$

Does this lead to a better comparison between model and experiment?

> $ms := .011;$ *# spring has mass 11g*
$M := m + \frac{1}{3} \cdot ms;$ *# "effective mass"*

$$ms := 0.011$$
$$M := 0.1036666667 \qquad (5)$$

> $\omega := \sqrt{\frac{k}{M}};$ *# predicted angular frequency*
$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right);$ *# predicted frequency*
$T := \frac{1}{f};$ *# predicted period*

$$\omega := 5.559883146$$
$$f := 0.8848828855$$
$$T := 1.130093051 \qquad (6)$$

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Wed Mar 22

Section 5.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems).

There are two methods we will use:

- The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 5.5.

- The method of variation of parameters is more general, and yields an integral formula for a particular solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f. It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$. We'll study the case $n = 2$ only.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in the notes.

Exercise 1) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3 .$$

Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10, b_2 = 3$.

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4y' - 5y = 10x + 3$$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x}.$$

Hint: try $y_P = d\,e^{2x}$ because L transforms functions of that form into ones of the form $b\,e^{2x}$, i.e. $L\left(d\,e^{2x}\right) = b\,e^{2x}$. "$d$" is your "undetermined coefficient" for $b = 14$.

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6\,.$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6$$
$$y(0) = 4$$
$$y'(0) = -4\,.$$

4c) Check your answer with technology.

```
> with(DEtools):
> dsolve({y''(x) + 4·y'(x) − 5·y(x) = 14·e^(2·x) − 20·x − 6, y(0) = 4, y'(0) = -4});
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$$y(x) = \frac{8}{5}\,e^{-5\,x} - 4\,e^{x} + 2\,e^{2\,x} + 4\,x + \frac{22}{5} \tag{7}$$

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Exercise 5) Find a particular solution to

$$L(y) := y'' + 4\,y' - 5\,y = 2\cos(3\,x) .$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \to V$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2\,x$.
- In Exercise 3 $V = span\{e^{2\,x}\}$ and so we guessed $y_P = d\,e^{2\,x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why $V = span\{\cos(3\,x)\}$ and a guess of $y_P = d\cos(3\,x)$ won't work?

```
> with(DEtools):
  dsolve(y''(x) + 4·y'(x) − 5·y(x) = 2·cos(3·x), y(x));
```

$$y(x) = e^{-5\,x}\_C2 + e^{x}\_C1 - \frac{7}{85}\cos(3\,x) + \frac{6}{85}\sin(3\,x) \qquad \textbf{(8)}$$

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All of the previous exercises rely on:

<u>Method of undetermined coefficients</u> (<u>base case</u>): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that
(i) $L : V \to V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

<u>why:</u> We already know this fact is true for matrix transformations $L(\underline{\mathbf{x}}) = A_{n \times n}\underline{\mathbf{x}}$ with $L : \mathbb{R}^n \to \mathbb{R}^n$ (because if the only homogeneous solution is $\underline{\mathbf{x}} = \underline{\mathbf{0}}$ then A reduces to the identity, so also each matrix equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$ has a unique solution $\underline{\mathbf{x}}$. The theorem above is a generalization of that fact to general linear transformations. There is an "appendix" explaining this at the end of today's notes, for students who'd like to understand the details.

<u>Exercise 6)</u> Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

(assuming the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

<u>6a)</u> $L(y) = x^3 + 6x - 5$

<u>6b)</u> $L(y) = 4e^{2x}\sin(3x)$

<u>6c)</u> $L(y) = x\cos(2x)$

```
> dsolve(y"(x) + 4·y'(x) − 5·y(x) = x·cos(2·x), y(x));
```

$$y(x) = e^{-5x}\_C2 + e^{x}\_C1 + \frac{1}{21025}(-1305x + 508)\cos(2x) + \frac{1}{21025}(1160x + 644)\sin(2x) \quad \textbf{(9)}$$

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BUT LOOK OUT

Exercise 7a) Find a particular solution to

$$y'' + 4\,y' - 5\,y = 4\,e^{x}\,.$$

Hint: since $y_H = c_1 e^{x} + c_2 e^{-5\,x}$, a guess of $y_P = a\,e^{x}$ will not work (and $span\{e^{x}\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\,x\,e^{x}$$

and factor $L = \mathrm{D}^2 + 4\,\mathrm{D} - 5\,I = [\mathrm{D} + 5\,I] \circ [\mathrm{D} - I]$.

7b) check work with technology

```
> with(DEtools):
> dsolve(y''(x) + 4·y'(x) − 5·y(x) = 4·e^x, y(x));
```

$$y(x) = \mathrm{e}^{-5\,x}\,\_C2 + \mathrm{e}^{x}\,\_C1 + \frac{2}{3}\,x\,\mathrm{e}^{x} \qquad \textbf{(10)}$$

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A vector space theorem like the one for the base case, except for $L : V \to W$, combined with our understanding of how to factor constant coefficient differential operators (as in lab you're working on this week) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in pages 341-346 of the text, and the table on page 346, reproduced here.

Method of undetermined coefficients (extended case): If L has a factor $(\mathrm{D} - r\,I)^s$ and $e^{r\,x}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s, as in Exercise 7. (If you understood the homework problem last week about factoring L into composition of terms like $(\mathrm{D} - r\,I)^s$, then you have an inkling of why this recipe works. If you didn't understand that last week problem, there's another one this week so you get a second chance. :-)) You may also need to use superposition, as in Exercise 4, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega\,x) + b_2 \sin(\omega\,x)$	$x^s\left(c_1 \cos(\omega\,x) + c_2 \sin(\omega\,x)\right)$	$r = \pm\, i\,\omega$
$e^{a\,x}\left(b_1 \cos(\omega\,x) + b_2 \sin(\omega\,x)\right)$	$x^s e^{a\,x}\left(c_1 \cos(\omega\,x) + c_2 \sin(\omega\,x)\right)$	$r = a \pm i\omega$
$b_0 e^{a\,x}$	$x^s c_0 e^{a\,x}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right)e^{a\,x}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)e^{a\,x}$	$r = a$

Exercise 8) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

8a) $y''' + 2y'' = x^2 + 6x$
(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

> $dsolve\left(y'''(x) + 2\cdot y''(x) = x^2 + 6\cdot x, y(x)\right);$

$$y(x) = \frac{1}{24}x^4 + \frac{5}{12}x^3 + \frac{1}{4}\mathrm{e}^{-2x}\_C1 - \frac{5}{8}x^2 + \_C2\,x + \_C3 \qquad \textbf{(11)}$$

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8b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$
(So the characteristic polynomial for $L(y) = 0$ is
$r^2 - 4r + 13) = (r-2)^2 + 9 = (r - 2 + 3i)(r - 2 - 3i)$.)

> $dsolve\left(y''(x) - 4\cdot y'(x) + 13\cdot y(x) = 4\cdot \mathrm{e}^{2\cdot x}\cdot\sin(3\cdot x), y(x)\right);$

$$y(x) = \mathrm{e}^{2x}\sin(3x)\,\_C2 + \mathrm{e}^{2x}\cos(3x)\,\_C1 - \frac{2}{3}\mathrm{e}^{2x}\cos(3x)\,x \qquad \textbf{(12)}$$

>

8c) $y'' + 5y' + 4y = 5\cos(2x) + 4\mathrm{e}^x + 5\mathrm{e}^{-x}$.
(So the characteristic polynomial for $L(y) = 0$ is $p(r) = r^2 + 5r + 4 = (r+4)(r+1)$.)

> $dsolve\left(y''(x) + 5\cdot y'(x) + 4\cdot y(x) = 5\cdot\cos(2\cdot x) + 4\cdot\mathrm{e}^x + 5\cdot\mathrm{e}^{-x}, y(x)\right);$

$$y(x) = \mathrm{e}^{-x}\,\_C2 + \mathrm{e}^{-4x}\,\_C1 + \frac{1}{2}\sin(2x) + \frac{2}{5}\mathrm{e}^x + \frac{5}{3}\mathrm{e}^{-x}x - \frac{5}{9}\mathrm{e}^{-x} \qquad \textbf{(13)}$$

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Variation of Parameters: This is an alternate method for finding particular solutions. Its advantage is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x),...y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + ... + p_1(x)y' + p_0(x)y = 0 .$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + ... + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x),...u_n(x)$ have derivatives satisfying the matrix equation

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \left[W\left(y_1, y_2,..., y_n\right)\right]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

where $\left[W\left(y_1, y_2,..., y_n\right)\right]$ is the Wronskian matrix.

Here's how to check this fact when $n = 2$: Write

$$y_p = y = u_1 y_1 + u_2 y_2 .$$

Thus

$$y' = u_1 y_1{}' + u_2 y_2{}' + \left(u_1{}' y_1 + u_2{}' y_2\right) .$$

Set

$$\left(u_1{}' y_1 + u_2{}' y_2\right) = 0.$$

Then

$$y'' = u_1 y_1{}'' + u_2 y_2{}'' + \left(u_1{}' y_1{}' + u_2{}' y_2{}'\right) .$$

Set

$$\left(u_1{}' y_1{}' + u_2{}' y_2{}'\right) = f.$$

Notice that the two $(...)$ equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{bmatrix} \begin{bmatrix} u_1{}' \\ u_2{}' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p. Under these conditions we compute

$$\begin{aligned} & p_0 \left[y = u_1 y_1 + u_2 y_2\right] \\ & + p_1 \left[y' = u_1 y_1{}' + u_2 y_2{}'\right] \\ & + 1 \left[y'' = u_1 y_1{}'' + u_2 y_2{}'' + f\right] \\ & \quad L(y) = u_1 L\left(y_1\right) + u_2 L\left(y_2\right) + f \\ & L(y) = 0 + 0 + f = f \end{aligned}$$

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." We will not discuss these in class, but I'll be happy to chat about the arguments with anyone who's interested, outside of class. They only use ideas we've talked about already, although they are abstract.

Theorem 0:

- Let V and W be vector spaces. Let V have dimension $n < \infty$ and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\boldsymbol{v}}_n\}$ be a basis for V.
- Let $L : V \rightarrow W$ be a linear transformation, i.e. $L(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = L(\underline{\boldsymbol{u}}) + L(\underline{\boldsymbol{v}})$ and $L(c\,\underline{\boldsymbol{u}}) = c\,L(\underline{\boldsymbol{u}})$ holds $\forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in V, c \in \mathbb{R}$.) Consider the range of L, i.e.

$$\begin{aligned} Range(L) &:= \{L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n) \in W, \textit{such that each } d_j \in \mathbb{R}\} \\ &= \{d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\boldsymbol{v}}_n) \in W, \textit{such that each } d_j \in \mathbb{R}\} \\ &= span\{L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ... L(\underline{\boldsymbol{v}}_n)\}. \end{aligned}$$

Then $Range(L)$ is $n - dimensional$ if and only if the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$.

proof:

(i) $\Leftarrow$: The only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$ implies $Range(L)$ is $n - dimensional$:

If we can show $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ... L(\underline{\boldsymbol{v}}_n)$ are linearly independent, then they will be a basis for $Range(L)$ and this subspace will have dimension n. So, consider the dependency equation:

$$d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}}.$$

Because L is a linear transformation, we can rewrite this equation as

$$L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}}.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n = \underline{\boldsymbol{0}}.$$

Since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\boldsymbol{v}}_n$ are a basis they are linearly independent, so $d_1 = d_2 = ... = d_n = 0$.

□

(ii) $\Rightarrow$: $Range(L)$ is $n - dimensional$ implies the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$: Since the range of L is $n - dimensional$, $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ... L(\underline{\boldsymbol{v}}_n)$ must be linearly independent. Now, let $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n$ be a homogeneous solution, $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$. In other words,

$$\begin{aligned} & L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}} \\ \Rightarrow\ & d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}} \\ \Rightarrow\ & d_1 = d_2 = ... = d_n = 0 \Rightarrow \underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}. \end{aligned}$$

□

Theorem 1 Let Let V and W be vector spaces, both with the same dimension $n < \infty$. Let $L : V \rightarrow W$ be a linear transformation. Let the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$. Then for each $\underline{\boldsymbol{w}} \in W$ there is a unique $\underline{\boldsymbol{v}} \in V$ with $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$.

proof: By Theorem 0, the dimension of $Range(L)$ is $n - dimensional$. Therefore it must be all of W. So for each $\underline{\boldsymbol{w}} \in W$ there is at least one $\boldsymbol{v}_P \in V$ with $L(\underline{\boldsymbol{v}}_P) = \underline{\boldsymbol{w}}$. But the general solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{v}}_P + \underline{\boldsymbol{v}}_H$, where $\underline{\boldsymbol{v}}_H$ is the general solution to the homogeneous equation. By assumption, $\underline{\boldsymbol{v}}_H = \underline{\boldsymbol{0}}$, so the particular solution is unique.

□

<u>Remark:</u> In the <u>base case</u> of undetermined coefficients, $W = V$. In the <u>extended case,</u> W is the space in which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s. This is because if L factors as

$$L = \left(D - r_1 I\right)^{k_1} \circ \left(D - r_2 I\right)^{k_2} \circ \ldots \circ \left(D - r_m I\right)^{k_m}$$

and if f is in a subspace W associated with the characteristic polynomial root r_m, then for $s = k_m$ the factor $\left(D - r_m I\right)^{k_m}$ of L will transform the space $V = x^s W$ back into W, and not transform any non-zero function in V into the zero function. And the other factors of L will then preserve W, also without transforming any non-zero elements to the zero function.

Math 2250-004
Friday Mar 24

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on Monday using these notes.
Overview for solutions $x(t)$ to

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)$$

using section 5.5 undetermined coefficients algorithms.

- undamped $(c = 0)$:
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

which has simple harmonic motion solutions $x_H(t) = C\cos(\omega_0 t - \alpha)$. So for the non-homongeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \Rightarrow x_P = A\cos(\omega\,t)$ because only even derivatives, we don't need $\sin(\omega\,t)$ terms !!

$$\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!
- $\omega = \omega_0$ $\qquad \Rightarrow x_P = t\left(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\right)$

$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega_0\,t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

....

$$m\,x'' + c\,x' + k\,x = F_0\cos(\omega\,t)$$

- damped $(c > 0)$: in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm\,i\,\omega$ when $c > 0$).
  - underdamped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)$.
  - critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)$.
  - over-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

• in all three damped cases on the previous page, $x_H(t) \to 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \to \infty$. And in these damped cases $x_P(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

• if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important.

forced undamped oscillations:

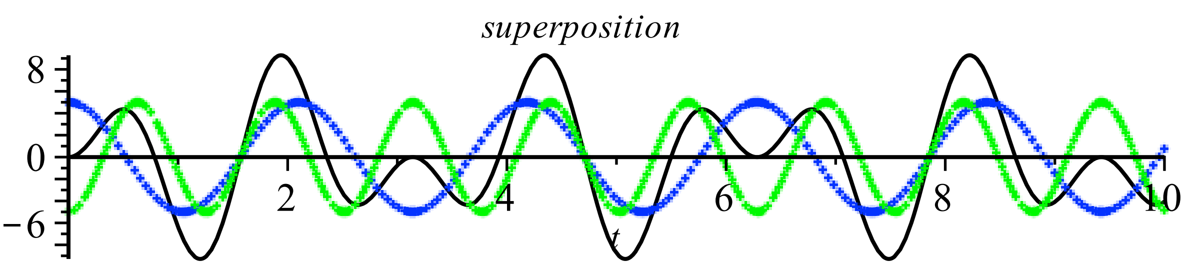
Exercise 1a) Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

```
> with(plots) :
> plot1 := plot(−5·cos(5·t), t = 0 ..10, color = green, style = point) :
  plot2 := plot(5·cos(3·t), t = 0 ..10, color = blue, style = point) :
  plot3 := plot(−5·cos(5·t) + 5·cos(3·t), t = 0 ..10, color = black) :
  display({plot1, plot2, plot3}, title='superposition');
```



In general:

undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$\begin{aligned} +\tfrac{k}{m}\,(& x_p = A \cos \omega t \ ) \\ +0\,(& x_p' = -A\omega \sin \omega t \ ) \\ +1\,(& x_p'' = -A\omega^2 \cos \omega t \ ) \end{aligned}$$

$$L(x_p) = \cos \omega t \; A \left[\tfrac{k}{m} - \omega^2 \right] \qquad \left(\tfrac{k}{m} = \omega_0^2\right)$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$

$$A = \frac{F_0}{m} \Big/ (\omega_0^2 - \omega^2)$$

So,
$x_p(t) = -\frac{F_0}{m} \frac{1}{(\omega^2 - \omega_0^2)} \cos \omega t$. Note $x_H(t) = A \cos \omega_0 t + B \sin \omega_0 t$.

so, by plugging in or observation
IVP solution is

$$x(t) = \frac{+F_0}{m(\omega^2 - \omega_0^2)} \left(\cos \omega_0 t - \cos \omega t \right) + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

check-WR!

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned} \cos(\alpha - \beta) - \cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\ &= 2 \sin(\alpha)\sin(\beta) \ . \end{aligned}$$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t$, $\beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2 \sin\left(\frac{1}{2}(\omega + \omega_0)t\right) \sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \qquad \text{period: } \frac{4\pi}{|\omega - \omega_0|} \ .$$

We will call half that period the beating period, as explained by the next exercise:

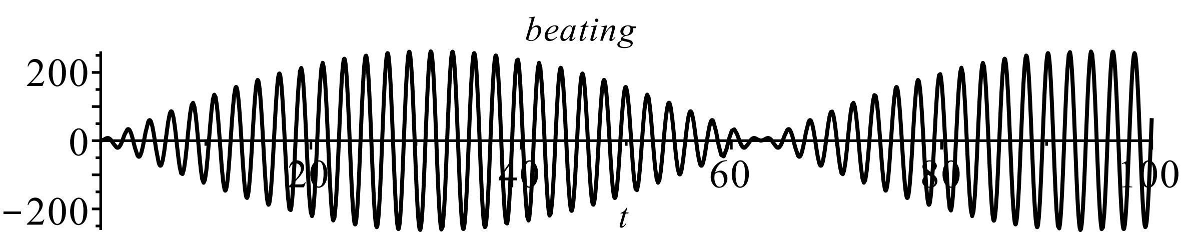
$$\text{beating period: } \frac{2\pi}{|\omega - \omega_0|}, \qquad \text{beating amplitude: } \frac{2 F_0}{m|\omega^2 - \omega_0^2|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

```
> plot(262.3·sin(3.05·t)sin(.05·t), t=0..100, color=black, title=`beating`);
```



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2 (\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad) \\
0 (\quad & x_p' = t\,(-A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad) \\
+1 (\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$x_p(t) = \underbrace{\frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}$$

sats $x(0)=0$, $x'(0)=0$, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class.

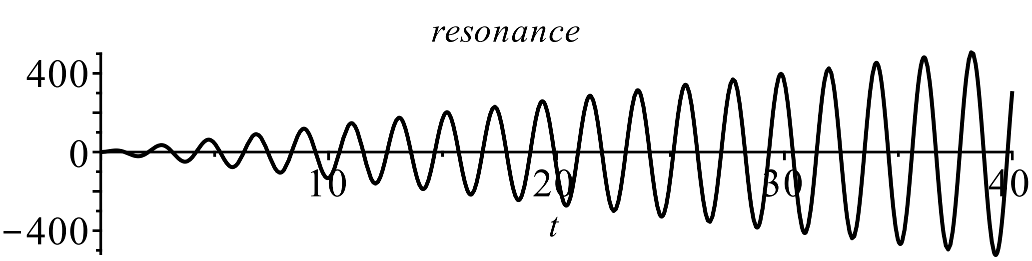
Exercise 3a) Solve the IVP

$$x'' + 9\,x = 80\cos(3\,t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t \cdot \sin(3 \cdot t),\ t = 0..40,\ color = black,\ title = \text{`resonance`}\right);$



>

• After finishing the discussion of undamped forced oscillations, we will discuss the physics and mathematics of damped forced oscillations

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)\ .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

Damped forced oscillations $(c > 0)$ for $x(t)$:

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k\,&\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\ +\,c\,&\left[x_P' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right] \\ +\,m\,&\left[x_P'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right] . \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega t)\left(k\,A + c\,B\,\omega - m\,A\,\omega^2\right) \\ &\quad + \sin(\omega t)\left(k\,B - c\,A\,\omega - m\,B\,\omega^2\right) . \end{aligned}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \quad . \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} .$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}(c_1 t + c_2)$.
- over-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

Exercise 4) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\omega^2 << \omega_0^2$ we have α near zero (in phase) for x_{sp}, because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have α near π (out of phase), because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1$, $\cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

Exercise 5) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0 .$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}\, e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3) .$$

```
> with(DEtools) :
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
```

$$x(t) = -3\,e^{-t}\sin(5t) + e^{-t}\cos(5t) + 4\sin(4t) + 5\cos(4t) \quad \textbf{(14)}$$

Practical resonance: The steady periodic amplitude C for damped forced oscillations is

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}.$$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$.

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 6a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

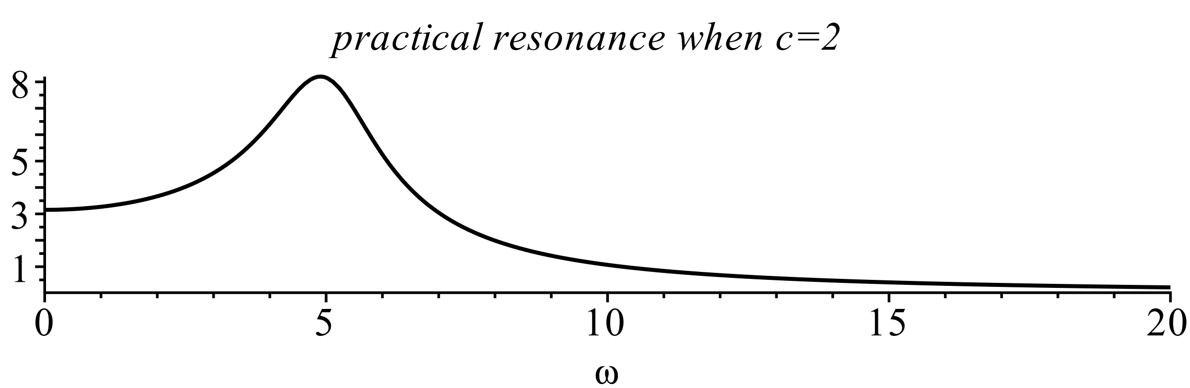
6b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

> *restart* :

> *with*(*plots*) :

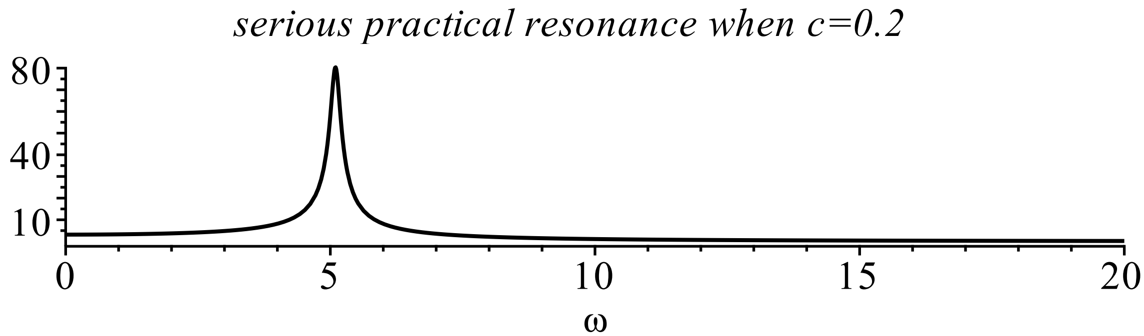
> $C := (\omega, c) \to \frac{82}{\sqrt{\left(26 - \omega^2\right)^2 + c^2 \cdot \omega^2}}$:

> *plot*(*C*(ω, 2), ω = 0 ..20, *color* = *black*, *title* = \`practical resonance when c=2\`);

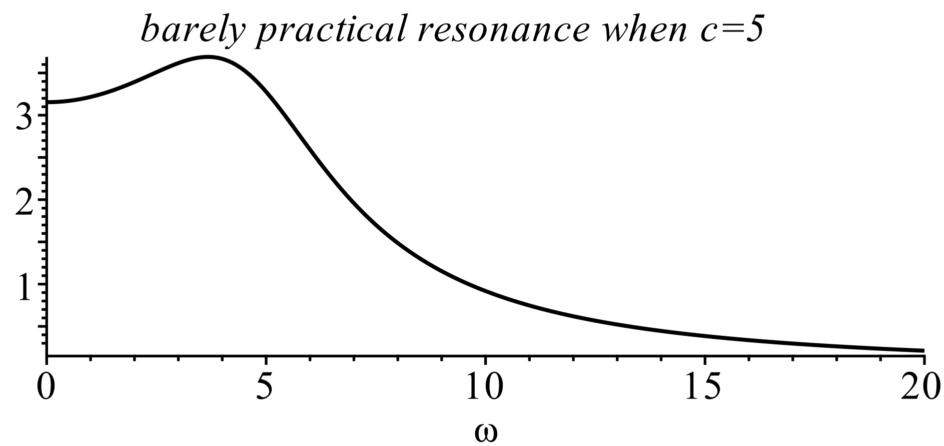


>

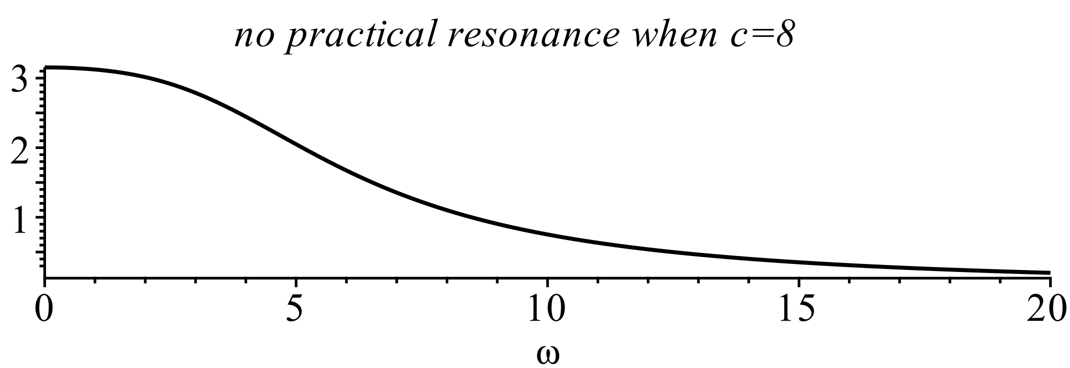
> $plot(C(\omega, .2), \omega = 0..20, color = black, title = \text{`serious practical resonance when c=0.2`});$



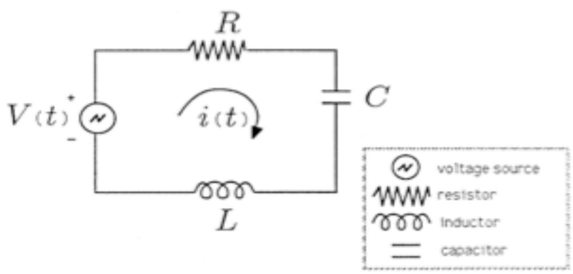
> $plot(C(\omega, 5), \omega = 0..20, color = black, title = \text{`barely practical resonance when c=5`});$



> $plot(C(\omega, 8), \omega = 0..20, color = black, title = \text{`no practical resonance when c=8`});$



The mechanical-electrical analogy, continued: Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....recall from earlier in the course:



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t) = E_0 \sin(\omega t)$

For $I(t)$: $L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t) = E_0\,\omega \cos(\omega t)$.

Transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t) .$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma) , \quad \gamma = \alpha - \frac{\pi}{2} .$$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \quad \Rightarrow I_0(\omega) = \frac{E_0\omega}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}} .$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the impedance $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\,\omega} = L\,\omega$, i.e.

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old radios).}$$

$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable}$$

Both L and C are adjusted in this M.I.T. lab demonstration:
http://www.youtube.com/watch?v=ZYgFuUl9_Vs.