Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing. It will help to know

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in Exercise 2 that we completed earlier. The theorem and the techniques we'll actually be using going forward are illustrated by

Exercise 5) Consider the homogeneous linear DE for $y(x)$

$$\boxed{L(y) = y'' - 2y' - 3y = 0}$$

$y' - ay = 0$

$y(x) = ce^{ax}$

5a) Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

5b) Show that every IVP

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = b_0 \\ y'(0) = b_1 \end{cases}$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.
Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation.
Next, show $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ are linearly independent by setting a linear combination equal to the zero function, differentiating that identity, and then substituting $x = 0$ into the resulting system of equations to deduce $c_1 = c_2 = 0$:

$$c_1 y_1(x) + c_2 y_2(x) = 0$$
$$\Rightarrow c_1 y_1'(x) + c_2 y_2'(x) = 0$$

so that $\{y_1(x) = e^{rx}, y_2(x) = e^{\rho x}\}$ is a basis for the the solution space. So also the solution space is two-dimensional since the basis consists of two functions.

5a) $y = e^{rx}$

$$L(e^{rx}) = r^2 e^{rx} - 2(re^{rx}) - 3e^{rx}$$
$$= e^{rx}[\underbrace{r^2 - 2r - 3}] = 0$$

"characteristic poly"
need roots

$r^2 - 2r - 3 = (r-3)(r+1) = 0$

$y_1(x) = e^{3x}$
$y_2(x) = e^{-x}$

$y_H(x) = c_1 e^{3x} + c_2 e^{-x}$
$y_H'(x) = 3c_1 e^{3x} - c_2 e^{-x}$

IVP @ $x = 0$

$y(0) = b_0 = c_1 + c_2$
$y'(0) = b_1 = 3c_1 - c_2$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

b) $\underline{\text{continued}}$: $\{y_1(x), y_2(x)\}$ are a $\underline{\text{basis}}$ for all solutions to

$$y'' - 2y' - 3y = 0$$

i) linear ind.

ii) span.

Let $y(x)$ solve the homog. DE.

solve

$$\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{for } c_1, c_2$$

then $c_1 y_1 + c_2 y_2$ matches $y(0), y'(0)$.

so by uniqueness theorem $y(x) = c_1 y_1 + c_2 y_2$ ∎

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix}$$

det $= -4 \neq 0$

W^{-1} exists.

so each IVP has a unique soln, given as $c_1 e^{3x} + c_2 e^{-x}$

$c_1 y_1 + c_2 y_2 = 0$ for all x.

$c_1 y_1' + c_2 y_2' = 0$ for all x.

@ $x = 0$ $\quad \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

5c) Now consider the inhomogeneous DE

$$L[y] = y'' - 2y' - 3y = 9$$

Notice that $y_p(x) = -3$ is a particular solution. Use this information and superposition (linearity) to solve the initial value problem

$$L(-3) = 0 + 0 - 3(-3) = 9 \checkmark$$

$$y'' - 2y' - 3y = 9$$
$$y(0) = 6$$
$$y'(0) = -2.$$

$$y = y_P + y_H$$

$$y = -3 + c_1 e^{3x} + c_2 e^{-x}$$

$$y' = 0 + 3c_1 e^{3x} - c_2 e^{-x}$$

@ $x=0$: $\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 \\ 29 \end{bmatrix} = \begin{bmatrix} 7/4 \\ 29/4 \end{bmatrix}$$

$$\boxed{y(x) = -3 + \frac{7}{4} e^{3x} + \frac{29}{4} e^{-x}}$$

$$y(0) = -3 + \frac{7}{4} + \frac{29}{4} = 6 \checkmark$$

$$y'(0) = \frac{21}{4} - \frac{29}{4} = -\frac{8}{4} = -2 \checkmark$$

yipes

Math 2250-004
$\boxed{\text{Fri Mar 3}}$ → $\boxed{\text{Tuesday}}$

2/3 of class "matrix magic" Chptr 4
1/3 Chptr 5

First, finish and/or review our $\boxed{\text{Wednesday}}$ discussions.

Although we don't have the tools yet to prove the existence-uniqueness result Theorem 2, we can use it to prove the dimension result Theorem 3. Here's how (and this is really just an abstractified version of Exercise 5 in Wednesday's notes).

Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on an interval I for which the hypotheses of the existence-uniqueness theorem hold.
Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to IVP's at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\boxed{\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by good guessing, as in Exercise 5 on Friday, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:
• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, $y' = c_1 y_1' + c_2 y_2'$ so to solve the IVP

$$y'' + p(x)y' + q(x)y = 0$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

we set

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$
$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$$

At x_0 we wish to find c_1, c_2 so that

$$c_1 y_1(x_0) + c_2 y_2(x_0) = b_0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = b_1$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Since the Wronskian matrix at x_0 has and inverse, the unique solution $[c_1, c_2]^T$ is given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, this must be it. Since each solution

$y(x)$ to the differential equation solves *some* initial value problem at x_0, each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus y_1, y_2 span the solution space.

• Linear independence: If we have the identity

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

then by differentiating each side with respect to x we also have

$$c_1 y_1{}'(x) + c_2 y_2{}'(x) = 0.$$

Evaluating at $x = x_0$ this is the system

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$
$$c_1 y_1{}'(x_0) + c_2 y_2{}'(x_0) = 0$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1{}'(x_0) & y_2{}'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad \checkmark.$$

5.2: general theory for n^{th}-order linear differential equations; tests for linear independence;
also begin 5.3: finding the solution space to homogeneous linear constant coefficient differential equations by trying exponential functions as potential basis functions.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$y''(x_0) = b_2$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

and to learn important physics/engineering applications of these general techniques.

The algorithm for solving these DEs and IVPs is:

(1) Find a basis $y_1, y_2, \ldots y_n$ for the $n-$dimensional homogeneous solution space, so that the general homogeneous solution is their span, i.e. $y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$.
(2) If the DE is non-homogeneous, find a particular solution y_P. Then the general solution to the non-homogeneous DE is $y = y_P + y_H$. (If the DE is homogeneous you can think of taking $y_P = 0$, since $y = y_H$.)
(3) Find values for the n free parameters $c_1, c_2, \ldots c_n$ in

$$y = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

to solve the initial value problem with initial values $b_0, b_1, \ldots, b_{n-1}$. (This last step just reduces to a matrix problem like in Chapter 3, where the matrix is the Wronskian matrix of $y_1, y_2, \ldots y_n$, evaluated at x_0 and the right hand side vector comes from the initial values and the particular solution and its derivatives' values at x_0.)

We've already been exploring how these steps play out in examples and homework problems, but will be studying them more systematically today and Monday. On Tuesday we'll begin the applications in section 5.4. We should have some fun experiments later next week to compare our mathematical modeling with physical reality.

Definition: An n^{th} order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = F(x) .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

($a_{n-1}, \ldots a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I.)

This DE is called linear because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(cy) = cL(y)$, $c \in \mathbb{R}$.

- *The proof that L satisfies the linearity proporties is just the same as it was for the case when n = 2, that we checked Wednesday. Then, since the $y = y_P + y_H$ proof only depended on the linearity properties of L, just like yesterday, we deduce both of Theorems 0 and 1:*

Theorem 0: The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

is a subspace.

Theorem 1: The general solution to the nonhomogeneous n^{th} order linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Later in the course we'll understand n^{th} order existence uniqueness theorems for initial value problems, in a way analogous to how we understood the first order theorem using slope fields, but let's postpone that discussion and just record the following true theorem as a fact:

Theorem 2 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x), \ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$
\begin{gathered}
y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\
y(x_0) = b_0 \\
y'(x_0) = b_1 \\
y''(x_0) = b_2 \\
\vdots \\
y^{(n-1)}(x_0) = b_{n-1}
\end{gathered}
$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

Just as for the case n = 2, the existence-uniqueness theorem lets you figure out the dimension of the solution space to homogeneous linear differential equations. The proof is conceptually the same, but messier to write down because the vectors and matrices are bigger.

Theorem 3: The solution space to the n^{th} order homogeneous linear differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, \ldots y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

proof: By the existence half of Theorem 2, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$
\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix},
\begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, \ldots,
\begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}
$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

Claim: In this case, the solutions $y_1, y_2, \ldots y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y(x) = z(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n .$$

Thus $y_1, y_2, \ldots y_n$ <u>span</u> the solution space.

- **<u>linear independence</u>**: If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then differentiate this identity $n-1$ times, and then substitute $x = x_0$ into the resulting n equations. This yields the Wronskian matrix equation above, with $\left[b_0, b_1, \ldots b_{n-1}\right]^T = [0, 0, \ldots, 0]^T$. So the matrix equation above implies that $\left[c_1, c_2, \ldots c_n\right]^T = \underline{\mathbf{0}}$. So $y_1, y_2, \ldots y_n$ are also <u>linearly independent</u>.

- Thus $y_1, y_2, \ldots y_n$ are a <u>basis</u> for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n .$$

□

Let's do some new exercises that tie these ideas together. (We may do these excersises while or before we wade through the general discussions on the previous pages!

Exercise 1) Consider the 3^{rd} order linear homogeneous DE for $y(x)$:

$$y''' + 3y'' - y' - 3y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Make sure to use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

try $e^{rx} = y$. $\Rightarrow y''' + 3y'' - y' - 3y = r^3 e^{rx} + 3r^2 e^{rx} - re^{rx} - 3e^{rx}$

$$= e^{rx}\underbrace{(r^3 + 3r^2 - r - 3)}_{p(r)} \underset{\text{want}}{=} 0 \quad \text{for all } x$$

$p(r)$ "characteristic polynomial"

we want its roots

$$\begin{aligned} &r^3 + 3r^2 - r - 3 \\ &= r^2(r+3) - (r+3) \\ &= (r^2-1)(r+3) \\ &= (r-1)(r+1)(r+3) = 0 \end{aligned}$$

roots $r = 1, -1, -3$.

soltns $y_1(x) = e^x,\ y_2(x) = e^{-x},\ y_3(x) = e^{-3x}$

$$W = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{-x} & e^{-3x} \\ e^x & -e^{-x} & -3e^{-3x} \\ e^x & e^{-x} & 9e^{-3x} \end{bmatrix}$$

@ $x=0$: $Wron = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 8 \end{vmatrix} \begin{matrix} \\ -R_1+R_2 \\ -R_1+R_3 \end{matrix}$$

$= -16 \neq 0$!

Exercise 2a) Find the general solution to

$$y''' + 3y'' - y' - 3y = 6\,.$$

Hint: First try to find a particular solution ... try a constant function.

$y = y_P + y_H$

$y_P = -2$ works!

$y_p = d$ cnts

$L(y_p) = 0 + 3\cdot 0 - 0 - 3d = 6$

$d = -2$ works.

$y = -2 + c_1 e^x + c_2 e^{-x} + c_3 e^{-3x}$

$y' = 0 + c_1 e^x - c_2 e^{-x} - 3c_3 e^{-3x}$

$y'' = 0 + c_1 e^x + c_2 e^{-x} + 9c_3 e^{-3x}$

b) Set up the linear system to solve the initial value problem for this DE, with $y(0) = -1, y'(0) = 2, y''(0) = 7$.

@ $x=0$: $\begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

there's that Wronskian matrix @ $x=0$, again

for fun now, but maybe not just for fun later:

```
> with(DEtools) :
  dsolve({y'''(x) + 3·y''(x) − y'(x) − 3·y(x) = 6, y(0) = −1, y'(0) = 2, y''(0) = 7});
```

$$y(x) = -2 + \frac{9}{4}e^x + \frac{3}{4}e^{-3x} - 2e^{-x} \tag{1}$$

Math 2250-004
Week 9: March 6-10 5.2-5.4

Mon Mar 6: Section 5.2 from last week.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$\text{IVP}\quad \begin{cases} y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ y''(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

$y' + p(x)y = q(x)$
Chptr 1.

<u>and</u> to learn important physics/engineering applications of these general techniques.

Finish Wednesday March 1 notes from last week which discuss the case $n = 2$, and continue into the Friday March 3 notes which discuss the analogous ideas for general n. This is section 5.2 of the text.

Hw: 5.2.16 $\quad y''' - 5y'' + 8y' - 4y = 0$

$$\left.\begin{aligned} y(0) &= 1 \\ y'(0) &= 4 \\ y''(0) &= 0 \end{aligned}\right\}$$

$y_1 = e^x$
$y_2 = e^{2x}$
$y_3 = xe^{2x}$

$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3$
$y(x) = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$
$y'(x) = c_1 e^x + 2c_2 e^{2x} + c_3 (e^{2x} + 2xe^{2x})$
$y''(x) = c_1 e^x + 4c_2 e^{2x} + c_3 (2e^{2x} + 2e^{2x} + 4xe^{2x})$

@ $x=0$: $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ Solve!

Tues Mar 7

5.3: Algorithms for the basis and general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \dots a_1, a_0$ are all constant.

step 1) Try to find a basis made of exponential functions....try $y(x) = e^{rx}$. In this case

$$L(y) = e^{rx}\left(r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0\right) = e^{rx}p(r).$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \dots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}.$$

Example: Exercise 1 from last Friday's notes: The differential equation

$$y''' + 3y'' - y' - 3y = 0$$

has characteristic polynomial

$$p(r) = r^3 + 3r^2 - r - 3 = (r+3)\cdot\left(r^2 - 1\right) = (r+3)(r+1)\cdot(r-1)$$

so the general solution to

$$y''' + 3y'' - y' - 3y = 0$$

is

$$y_H(x) = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + c_3 \mathrm{e}^{-3x}.$$

Exercise 1) By construction, $e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \dots < r_n$ and to use a limiting argument.