Example $f_1(x) = 1,\quad f_2(x) = x,\quad f_3(x) = x^2$

$$\text{span}\{f_1, f_2, f_3\} = \{af_1 + bf_2 + cf_3,\ a,b,c \in \mathbb{R}\}$$
$$= \{f(x) = a\cdot 1 + b\cdot x + c\cdot x^2,\ a,b,c \in \mathbb{R}\}$$
$$= \text{space of polynomials of degree} \le 2.$$

are f_1, f_2, f_3 linearly ind.

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \leftarrow \text{zero function}$$

at each x $\quad \cancel{c_1} + c_2 x + c_3 x^2 = 0$

@ $x=0$: $\quad c_1 = 0$

@ $x=1$: $\quad c_2 + c_3 = 0$

@ $x=-1$: $\quad -c_2 + c_3 = 0$

$\Rightarrow c_2, c_3 = 0$

so $\{1, x, x^2\}$ is linearly independent.

another way:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all } x$$

D_x: $\quad c_1 f_1'(x) + c_2 f_2'(x) + c_3 f_3'(x) = 0 \quad \forall x$

D_x: $\quad c_1 f_1''(x) + c_2 f_2''(x) + c_3 f_3''(x) = 0 \quad \forall x$

Wronskian matrix of f_1, f_2, f_3 $\rightarrow$

$$\begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{holds for all } x$$

$$\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{holds for all } x$$

@ $x=0$ $\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0.$

Definition: A second order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$\boxed{y'' + p(x)y' + q(x)y = f(x).}$$ vs. $y' + p(x)y = q(x)$

One reason this DE is called linear is that the "operator" L defined by

$$\boxed{L(y) := y'' + p(x)y' + q(x)y}$$

$L(\vec{u}) := A(\vec{u})$

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

$A(c\vec{u}) = cA\vec{u}$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(cy) = cL(y)$, $c \in \mathbb{R}$.

(Recall that the matrix multiplication function $L(\underline{\mathbf{x}}) := A\,\underline{\mathbf{x}}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a linear transformation.)

Exercise 1a) Check the linearity properties (1),(2) for the differential operator L.

(1) $L(y_1+y_2) = (y_1+y_2)'' + p\cdot(y_1+y_2)' + q(y_1+y_2)$

$= y_1'' + y_2'' + p\cdot(y_1' + y_2') + q\cdot(y_1+y_2)$

$= y_1'' + py_1' + qy_1 + y_2'' + py_2' + qy_2 = L(y_1) + L(y_2)$ ✓

1b) Use these properties to show that

(2) $L(cy) = (cy)'' + p\cdot(cy)' + q(cy)$

$= c(y'' + py' + qy) = cL(y)$

Theorem 0: the solution space to the homogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the "same" proof we used Monday to show that the solution space to a homogeneous matrix equation is a subspace.

α) If $L(y_1)=0$, $L(y_2)=0$

then $L(y_1+y_2) = L(y_1) + L(y_2) = 0$

β) If $L(y) = 0$ then $L(cy) = cL(y) = c\cdot 0 = 0$

soln space to $A\vec{x} = \vec{0}$ is subspace

if $A(\vec{u}) = \vec{0}$, $A(\vec{v}) = 0$

(α) then $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0}$

(β) $A(c\vec{u}) = cA(\vec{u}) = c\vec{0} = \vec{0}$.

Exercise 2) Find the solution space to homogeneous differential equation for $y(x)$

$$\boxed{y'' + 2y' = 0}$$ implicit description.

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two functions. Hint: This is really a first order DE for $v = y'$.

example

$L(y) = y'' + 2y'$

$L(e^{3x}) = 9e^{3x} + 2(3e^{3x})$

$= 15e^{3x}$

$L(2x+3) = 0 + 2(2) = 4$

$L(e^{-2x}) = 4e^{-2x} + 2(-2e^{-2x})$

$= 0$!

$v' + 2v = 0$

$e^{2x}(v' + 2v) = 0\cdot e^{2x} = 0$

$\frac{d}{dx}(e^{2x}v) = 0$

$e^{2x}v = C$

$\Rightarrow v = Ce^{-2x}$

$y' = Ce^{-2x}$

$\Rightarrow y = \int Ce^{-2x}dx = \frac{C}{-2}e^{-2x} + D$

$\boxed{y(x) = c_1e^{-2x} + c_2}$ explicit description

soln space to $y'' + 2y' = 0$

$= \text{span}\{e^{-2x}, 1\}$

Exercise 3) Use the linearity properties to show

Theorem 1: All solutions to the nonhomogeneous second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = f(x)$$

are of the form $\boxed{y = y_P + y_H}$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE. (You had a homework problem related to this idea, 3.4.40, in homework 5, but in the context of matrix equations, a week or two ago. The same idea reappears in this week's lab 6, in problem 1d !)

Let $L(y_p) = f$

Let $L(y_H) = 0$

Then $L(y_p + y_H) = L(y_p) + L(y_H) = f + 0 = f$

Let $L(y_Q) = f$ be another solution

write $y_Q = y_p + \underbrace{(y_Q - y_p)}$

$$L(y_Q - y_p) = L(y_Q) - L(y_p) = f - f = 0$$

So $y_Q - y_p$ is a homogeneous soltn.

if $A\vec{x} = \vec{b}$

and $A\vec{x}_1 = \vec{0}$

then $A(\vec{x} + \vec{x}_1) = A\vec{x} + A\vec{x}_1 = \vec{b} + \vec{0} = \vec{b}$

and if $A\vec{x}_2 = \vec{b}$

then $\vec{x}_2 = \vec{x} + \underbrace{(\vec{x}_2 - \vec{x})}_{\text{“}\vec{x}_1\text{”}}$

and $A(\vec{x}_2 - \vec{x}) = \vec{b} - \vec{b} = \vec{0}$

Exercise 4) Verify Theorem 1 for the differential equation

$$y'' + 2y' = 3$$

$y' + p(x)y = q(x)$

$p(x) = 2$

I.F. $e^{\int 2\,dx} = e^{2x}$

$v = y'$: $\quad v' + 2v = 3$

$$e^{2x}(v' + 2v) = 3e^{2x}$$

$$\frac{d}{dx}(e^{2x}v) = 3e^{2x}$$

$$e^{2x}v = \int 3e^{2x}\,dx$$

$$e^{2x}v = \frac{3}{2}e^{2x} + C$$

$$v = \frac{3}{2} + Ce^{-2x}$$

$$y' = \frac{3}{2} + Ce^{-2x}$$

$$\Rightarrow y(x) = \frac{3}{2}x + \frac{C}{-2}e^{-2x} + D$$

$$y(x) = \underbrace{\frac{3}{2}x}_{y_p(x)} + \underbrace{c_1 e^{-2x} + c_2}_{y_H(x)}$$

$$L\left(\frac{3}{2}x\right) = 0 + 2\left(\frac{3}{2}\right) = 3 \quad \checkmark$$

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 5) Verify Theorem 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 3$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

$$y(x) = \frac{3}{2}x + c_1 e^{-2x} + c_2$$
$$y'(x) = \frac{3}{2} - 2c_1 e^{-2x} + 0$$

$$@\, x=0: \quad 0 + c_1 + c_2 = b_0$$
$$\frac{3}{2} - 2c_1 = b_1$$

$$\begin{bmatrix} 0 \\ 3/2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}}_{\begin{bmatrix} e^{-2x} & 1 \\ -2e^{-2x} & 0 \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$@\, x=0$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 - 3/2 \end{bmatrix}$$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing. It will help to know

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in Exercise 2 that we completed earlier. The theorem and the techniques we'll actually be using going forward are illustrated by

Exercise 5) Consider the homogeneous linear DE for $y(x)$

$$L(y) = y'' - 2y' - 3y = 0$$

$y' - ay = 0$

$y(x) = ce^{ax}$

5a) Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

5b) Show that every IVP

$$y'' - 2y' - 3y = 0$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation.

Next, show $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ are linearly independent by setting a linear combination equal to the zero function, differentiating that identity, and then substituting $x = 0$ into the resulting system of equations to deduce $c_1 = c_2 = 0$:

$$c_1 y_1(x) + c_2 y_2(x) = 0$$
$$\Rightarrow c_1 y_1'(x) + c_2 y_2'(x) = 0$$

so that $\{y_1(x) = e^{rx}, y_2(x) = e^{\rho x}\}$ is a basis for the the solution space. So also the solution space is two-dimensional since the basis consists of two functions.

a) $y = e^{rx}$

$$L(e^{rx}) = r^2 e^{rx} - 2(re^{rx}) - 3e^{rx}$$
$$= e^{rx}\underbrace{[r^2 - 2r - 3]}_{\text{"characteristic poly" need roots}} = 0$$

$r^2 - 2r - 3 = (r-3)(r+1) = 0$

$y_1(x) = e^{3x}$

$y_2(x) = e^{-x}$

$y_H(x) = c_1 e^{3x} + c_2 e^{-x}$

5c) Now consider the inhomogeneous DE

$$y'' - 2y' - 3y = 9$$

Notice that $y_p(x) = -3$ is a particular solution. Use this information and superposition (linearity) to solve the initial value problem

$$y'' - 2y' - 3y = 9$$
$$y(0) = 6$$
$$y'(0) = -2.$$

Math 2250-004
Week 9: March 6-10 5.2-5.4

Mon Mar 6: Section 5.2 from last week.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$\text{IVP}\quad\begin{cases} y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ y''(x_0) = b_2 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{cases}$$

$y' + p(x)y = q(x)$
Chptr 1.

<u>and</u> to learn important physics/engineering applications of these general techniques.

Finish Wednesday March 1 notes from last week which discuss the case $n = 2$, and continue into the Friday March 3 notes which discuss the analogous ideas for general n. This is section 5.2 of the text.