Math 2250-004
Wed Mar 1

• Wed: Mostly Tuesday's notes
• Quiz day!
• Lab tomorrow → function spaces as vector spaces

5.1 Second order linear differential equations, and vector space theory connections.

Definition: A vector space is a collection of objects together with and "addition" operation "+", and a scalar multiplication operation, so that the rules below all hold.

- (α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
- (β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

} subspace properties

$(f+g)' = f' + g'$
$(cf)' = cf'$

As well as:
(a) $f + g = g + f$ (commutative property)
(b) $f + (g + h) = (f + g) + h$ (associative property)
(c) $\exists\ 0 \in V$ so that $f + 0 = f$ is always true.
(d) $\forall\ f \in V\ \exists -f \in V$ so that $f + (-f) = 0$ (additive inverses)
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $(c_1 + c_2) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot (c_2 \cdot f) = (c_1 c_2) \cdot f$ (associative property)
(h) $1 \cdot f = f,\ (-1) \cdot f = -f,\ 0 \cdot f = 0$ (these last two actually follow from the others).

Examples we've seen:
(1) $\mathbb{R}^m$, with the usual vector addition and scalar multiplication, defined component-wise
(2) subspaces W of $\mathbb{R}^m$, which satisfy (α),(β), and therefore automatically satisfy (a)-(h), because the vectors in W also lie in $\mathbb{R}^m$.

Exercise 0) In Chapter 5 we focus on the vector space

$$V = C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R}\ s.t.\ f \text{ is a continuous function}\}$$

and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the function $f + g$ is defined by $(f + g)(x) := f(x) + g(x)$ and the scalar multiple $c f(x)$ is defined by $(c f)(x) := c f(x)$. What is the zero vector for functions?

Because the vector space axioms are exactly the arithmetic rules we used to work with linear combination equations, all of the concepts and vector space theorems we talked about for $\mathbb{R}^m$ and its subspaces make sense for the function vector space V and its subspaces. In particular we can talk about

- the span of a finite collection of functions $f_1, f_2, \ldots f_n$.
- linear independence/dependence for a collection of functions $f_1, f_2, \ldots f_n$.
- subspaces of V
- bases and dimension for finite dimensional subspaces. (The function space V itself is infinite dimensional, meaning that no finite collection of functions spans it.)

Example $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2$

$$\text{span}\{f_1, f_2, f_3\} = \{af_1 + bf_2 + cf_3, \ a,b,c \in \mathbb{R}\}$$
$$= \{f(x) = a\cdot 1 + b\cdot x + c\cdot x^2, \ a,b,c \in \mathbb{R}\}$$
$$= \text{space of polynomials of degree} \leq 2.$$

are f_1, f_2, f_3 linearly ind.

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \quad \leftarrow \text{zero function}$$

at each x $\quad c_1 + c_2 x + c_3 x^2 = 0$

@ $x=0$: $\quad c_1 = 0$

@ $x=1$: $\quad c_2 + c_3 = 0$

@ $x=-1$: $\quad -c_2 + c_3 = 0$ $\quad \Rightarrow c_2, c_3 = 0$

so $\{1, x, x^2\}$ is linearly independent.

another way:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0 \quad \text{for all } x$$

D_x: $\quad c_1 f_1'(x) + c_2 f_2'(x) + c_3 f_3'(x) = 0 \quad \forall x$

D_x: $\quad c_1 f_1''(x) + c_2 f_2''(x) + c_3 f_3''(x) = 0 \quad \forall x$

Wronskian matrix of f_1, f_2, f_3 $\longrightarrow$

$$\begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{holds for all } x$$

$$\begin{bmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \text{holds for all } x$$

@ $x=0$ $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = c_3 = 0.$$

Definition: A second order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y'' + p(x)y' + q(x)y = f(x).$$

vs. $y' + p(x)y = q(x)$

One reason this DE is called linear is that the "operator" L defined by

$$L(y) := y'' + p(x)y' + q(x)y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(cy) = cL(y)$, $c \in \mathbb{R}$.

$L(\vec{u}) := A(\vec{u})$

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

$A(c\vec{u}) = cA\vec{u}$

(Recall that the matrix multiplication function $L(\underline{x}) := A\underline{x}$ satisfies the analogous properties. Any time we have have a transformation L satisfying (1),(2), we say it is a linear transformation.)

Exercise 1a) Check the linearity properties (1),(2) for the differential operator L.

(1) $L(y_1+y_2) = (y_1+y_2)'' + p\cdot(y_1+y_2)' + q(y_1+y_2)$
$= y_1'' + y_2'' + p\cdot(y_1'+y_2') + q\cdot(y_1+y_2)$
$= y_1'' + py_1' + qy_1 + y_2'' + py_2' + qy_2 = L(y_1) + L(y_2)$ ✓

1b) Use these properties to show that

(2) $L(cy) = (cy)'' + p\cdot(cy)' + q(cy)$
$= c(y'' + py' + qy) = cL(y)$

Theorem 0: the solution space to the homogeneous second order linear DE

$$y'' + p(x)y' + q(x)y = 0$$

is a subspace. Notice that this is the "same" proof we used Monday to show that the solution space to a homogeneous matrix equation is a subspace.

α) If $L(y_1)=0$, $L(y_2)=0$
then $L(y_1+y_2) = L(y_1) + L(y_2) = 0$

β) If $L(y)=0$ then $L(cy) = cL(y) = c\cdot 0 = 0$

soln space to $A\vec{x}=\vec{0}$ is subspace
if $A(\vec{u})=\vec{0}$, $A(\vec{v})=0$
(α) then $A(\vec{u}+\vec{v}) = A\vec{u}+A\vec{v} = \vec{0}+\vec{0}$
(β) $A(c\vec{u}) = cA(\vec{u}) = c\vec{0} = \vec{0}$.

Exercise 2) Find the solution space to homogeneous differential equation for $y(x)$

$$y'' + 2y' = 0$$

implicit description.

on the x-interval $-\infty < x < \infty$. Notice that the solution space is the span of two functions. Hint: This is really a first order DE for $v = y'$.

$v' + 2v = 0$
$e^{2x}(v' + 2v) = 0\cdot e^{2x} = 0$
$\frac{d}{dx}(e^{2x}v) = 0$
$e^{2x}v = C$
$\Rightarrow v = Ce^{-2x}$
$y' = Ce^{-2x}$
$\Rightarrow y = \int Ce^{-2x}dx = \frac{C}{-2}e^{-2x} + D$

$y(x) = c_1e^{-2x} + c_2$ explicit description

example
$L(y) = y'' + 2y'$
$L(e^{3x}) = 9e^{3x} + 2(3e^{3x}) = 15e^{3x}$
$L(2x+3) = 0 + 2(2) = 4$
$L(e^{-2x}) = 4e^{-2x} + 2(-2e^{-2x}) = 0$!

soln space to $y''+2y'=0$ $= \text{span}\{e^{-2x}, 1\}$

Exercise 3) Use the linearity properties to show

Theorem 1: All solutions to the nonhomogeneous second order linear DE

$$L(y) = y'' + p(x)y' + q(x)y = f(x)$$

are of the form $y = y_P + y_H$ where y_P is any single particular solution and y_H is some solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text). Thus, if you can find a single particular solution to the nonhomogeneous DE, and all solutions to the homogeneous DE, you've actually found all solutions to the nonhomogeneous DE. (You had a homework problem related to this idea, 3.4.40, in homework 5, but in the context of matrix equations, a week or two ago. The same idea reappears in this week's lab 6, in problem 1d !)

Let $L(y_p) = f$

Let $L(y_H) = 0$

Then $L(y_p + y_H) = L(y_p) + L(y_H) = f + 0 = f$

Let $L(y_Q) = f$ be another solution

write $y_Q = y_p + \underbrace{(y_Q - y_p)}$

$L(y_Q - y_p) = L(y_Q) - L(y_p)$
$= f - f = 0$

so $y_Q - y_p$ is a homogeneous soln.

if $A\vec{x} = \vec{b}$

and $A\vec{x}_1 = \vec{0}$

then $A(\vec{x} + \vec{x}_1) = A\vec{x} + A\vec{x}_1$
$= \vec{b} + \vec{0}$
$= \vec{b}$

and if $A\vec{x}_2 = \vec{b}$

then $\vec{x}_2 = \vec{x} + \underbrace{(\vec{x}_2 - \vec{x})}_{\text{"}\vec{x}_1\text{"}}$

and $A(\vec{x}_2 - \vec{x})$
$= \vec{b} - \vec{b} = \vec{0}$

Exercise 4) Verify Theorem 1 for the differential equation

$$y'' + 2y' = 3$$

$y' + p(x)y = q(x)$

$p(x) = 2$

I.F. $e^{\int 2\,dx} = e^{2x}$

$v = y'$: $v' + 2v = 3$

$e^{2x}(v' + 2v) = 3e^{2x}$

$\frac{d}{dx}(e^{2x}v) = 3e^{2x}$

$e^{2x}v = \int 3e^{2x}\,dx$

$\rightarrow e^{2x}v = \frac{3}{2}e^{2x} + C$

$v = \frac{3}{2} + Ce^{-2x}$

$y' = \frac{3}{2} + Ce^{-2x}$

$\Rightarrow y(x) = \frac{3}{2}x + \frac{C}{-2}e^{-2x} + D$

$y(x) = \underbrace{\frac{3}{2}x}_{y_P(x)} + \underbrace{c_1 e^{-2x} + c_2}_{y_H(x)}$

$L(\frac{3}{2}x) = 0 + 2(\frac{3}{2}) = 3$ ✓

Theorem 2 (Existence-Uniqueness Theorem): Let $p(x), q(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$y'' + p(x)y' + q(x)y = f(x)$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$

and $y(x)$ exists and is twice continuously differentiable on the entire interval I.

Exercise 5) Verify Theorem 2 for the interval $I = (-\infty, \infty)$ and the IVP

$$y'' + 2y' = 3$$
$$y(0) = b_0$$
$$y'(0) = b_1$$

$$y(x) = \frac{3}{2}x + c_1 e^{-2x} + c_2$$
$$y'(x) = \frac{3}{2} - 2c_1 e^{-2x} + 0$$

@ $x=0$: $0 + c_1 + c_2 = b_0$
$\quad 3/2 - 2c_1 = b_1$

$$\begin{bmatrix} 0 \\ 3/2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\begin{bmatrix} e^{-2x} & 1 \\ -2e^{-2x} & 0 \end{bmatrix}$$

@ $x=0$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} - \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} b_0 \\ b_1 - 3/2 \end{bmatrix}$$

Unlike in the previous example, and unlike what was true for the first order linear differential equation

$$y' + p(x)y = q(x)$$

there is not a clever integrating factor formula that will always work to find the general solution of the second order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Rather, we will usually resort to vector space theory and algorithms based on clever guessing. It will help to know

Theorem 3: The solution space to the second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is 2-dimensional.

This Theorem is illustrated in Exercise 2 that we completed earlier. The theorem and the techniques we'll actually be using going forward are illustrated by

Exercise 5) Consider the homogeneous linear DE for $y(x)$

$$L(y) = y'' - 2y' - 3y = 0$$

$y' - ay = 0$
$y(x) = ce^{ax}$

5a) Find two exponential functions $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ that solve this DE.

5b) Show that every IVP

$$\begin{cases} y'' - 2y' - 3y = 0 \\ y(0) = b_0 \\ y'(0) = b_1 \end{cases}$$

can be solved with a unique linear combination $y(x) = c_1 y_1(x) + c_2 y_2(x)$.
Then use the uniqueness theorem to deduce that y_1, y_2 span the solution space to this homogeneous differential equation.
Next, show $y_1(x) = e^{rx}$, $y_2(x) = e^{\rho x}$ are linearly independent by setting a linear combination equal to the zero function, differentiating that identity, and then substituting $x = 0$ into the resulting system of equations to deduce $c_1 = c_2 = 0$:

$$c_1 y_1(x) + c_2 y_2(x) = 0$$
$$\Rightarrow c_1 y_1'(x) + c_2 y_2'(x) = 0$$

so that $\{y_1(x) = e^{rx}, y_2(x) = e^{\rho x}\}$ is a basis for the the solution space. So also the solution space is two-dimensional since the basis consists of two functions.

5a) $y = e^{rx}$ $\quad L(e^{rx}) = r^2 e^{rx} - 2(re^{rx}) - 3e^{rx}$

$= e^{rx}[r^2 - 2r - 3] = 0$

"characteristic poly"
need roots

$r^2 - 2r - 3 = (r-3)(r+1) = 0$

$y_1(x) = e^{3x}$
$y_2(x) = e^{-x}$

$y_H(x) = c_1 e^{3x} + c_2 e^{-x}$
$y_H'(x) = 3c_1 e^{3x} - c_2 e^{-x}$

IVP @ $x = 0$

$y(0) = b_0 = c_1 + c_2$
$y'(0) = b_1 = 3c_1 - c_2$

$$\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

b) continued : $\{y_1(x), y_2(x)\}$ are a basis for all solutions to

$$y'' - 2y' - 3y = 0$$

i) linear ind.

ii) span.

Let $y(x)$ solve the homog. DE.

solve

$$\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ for c_1, c_2

then $c_1 y_1 + c_2 y_2$ matches $y(0), y'(0)$.

so by uniqueness theorem $y(x) = c_1 y_1 + c_2 y_2$ ∎

$c_1 y_1 + c_2 y_2 = 0$ for all x.

$c_1 y_1' + c_2 y_2' = 0$ for all x.

@ $x = 0$ $$\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix}$$

det $= -4 \neq 0$

W^{-1} exists.

so each IVP has a unique solution, given as $c_1 e^{3x} + c_2 e^{-x}$

5c) Now consider the inhomogeneous DE

$$L[y] = y'' - 2y' - 3y = 9$$

Notice that $y_P(x) = -3$ is a particular solution. Use this information and superposition (linearity) to solve the initial value problem

$$L(-3) = 0 + 0 - 3(-3) = 9 \checkmark$$

$$y'' - 2y' - 3y = 9$$
$$y(0) = 6$$
$$y'(0) = -2.$$

$y = y_P + y_H$

$y = -3 + c_1 e^{3x} + c_2 e^{-x}$

$y' = 0 + 3c_1 e^{3x} - c_2 e^{-x}$

@ $x=0$: $\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\begin{bmatrix} 9 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ -2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 7 \\ 29 \end{bmatrix} = \begin{bmatrix} 7/4 \\ 29/4 \end{bmatrix}$$

$$\boxed{y(x) = -3 + \frac{7}{4} e^{3x} + \frac{29}{4} e^{-x}}$$

$y(0) = -3 + \frac{7}{4} + \frac{29}{4} = 6 \checkmark$

$y'(0) = \frac{21}{4} - \frac{29}{4} = -\frac{8}{4} = -2 \checkmark$

yipes

Math 2250-004
Fri Mar 3 → Tuesday — 2/3 of class "matrix magic" Chptr 4; 1/3 Chptr 5

First, finish and/or review our Wednesday discussions.

Although we don't have the tools yet to prove the existence-uniqueness result Theorem 2, we can use it to prove the dimension result Theorem 3. Here's how (and this is really just an abstractified version of Exercise 5 in Wednesday's notes).

Consider the homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0$$

on an interval I for which the hypotheses of the existence-uniqueness theorem hold.
Pick any $x_0 \in I$. Find solutions $y_1(x), y_2(x)$ to IVP's at x_0 so that the so-called Wronskian matrix for y_1, y_2 at x_0

$$W(y_1, y_2)(x_0) = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}$$

is invertible (i.e. $\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \end{bmatrix}, \begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \end{bmatrix}$ are a basis for $\mathbb{R}^2$, or equivalently so that the determinant of the Wronskian matrix (called just the Wronskian) is non-zero at x_0).

- You may be able to find suitable y_1, y_2 by good guessing, as in Exercise 5 on Friday, but the existence-uniqueness theorem guarantees they exist even if you don't know how to find formulas for them.

Under these conditions, the solutions y_1, y_2 are actually a basis for the solution space! Here's why:
• span: the condition that the Wronskian matrix is invertible at x_0 means we can solve each IVP there with a linear combination $y = c_1 y_1 + c_2 y_2$: In that case, $y' = c_1 y_1' + c_2 y_2'$ so to solve the IVP

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(x_0) &= b_0 \\ y'(x_0) &= b_1 \end{aligned}$$

we set

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$
$$y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$$

At x_0 we wish to find c_1, c_2 so that

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= b_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= b_1 \end{aligned}$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Since the Wronskian matrix at x_0 has and inverse, the unique solution $[c_1, c_2]^T$ is given by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}.$$

Since the uniqueness theorem says each IVP has a unique solution, this must be it. Since each solution

$y(x)$ to the differential equation solves *some* initial value problem at x_0, each solution $y(x)$ *is* a linear combination of y_1, y_2. Thus y_1, y_2 <u>span</u> the solution space.

• <u>Linear independence:</u> If we have the identity

$$c_1 y_1(x) + c_2 y_2(x) = 0$$

then by differentiating each side with respect to x we also have

$$c_1 y_1'(x) + c_2 y_2'(x) = 0.$$

Evaluating at $x = x_0$ this is the system

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ ✓.

5.2: general theory for n^{th}-order linear differential equations; tests for linear independence;
also begin 5.3: finding the solution space to homogeneous linear constant coefficient differential equations by trying exponential functions as potential basis functions.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$
$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$y''(x_0) = b_2$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

and to learn important physics/engineering applications of these general techniques.

The algorithm for solving these DEs and IVPs is:

(1) Find a basis $y_1, y_2, \ldots y_n$ for the n-dimensional homogeneous solution space, so that the general homogeneous solution is their span, i.e. $y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$.
(2) If the DE is non-homogeneous, find a particular solution y_P. Then the general solution to the non-homogeneous DE is $y = y_P + y_H$. (If the DE is homogeneous you can think of taking $y_P = 0$, since $y = y_H$.)
(3) Find values for the n free parameters $c_1, c_2, \ldots c_n$ in

$$y = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

to solve the initial value problem with initial values $b_0, b_1, \ldots, b_{n-1}$. (This last step just reduces to a matrix problem like in Chapter 3, where the matrix is the Wronskian matrix of $y_1, y_2, \ldots y_n$, evaluated at x_0 and the right hand side vector comes from the initial values and the particular solution and its derivatives' values at x_0.)

We've already been exploring how these steps play out in examples and homework problems, but will be studying them more systematically today and Monday. On Tuesday we'll begin the applications in section 5.4. We should have some fun experiments later next week to compare our mathematical modeling with physical reality.

Definition: An n^{th} order linear differential equation for a function $y(x)$ is a differential equation that can be written in the form

$$A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_1(x)y' + A_0(x)y = F(x)\ .$$

We search for solution functions $y(x)$ defined on some specified interval I of the form $a < x < b$, or (a, ∞), $(-\infty, a)$ or (usually) the entire real line $(-\infty, \infty)$. In this chapter we assume the function $A_n(x) \neq 0$ on I, and divide by it in order to rewrite the differential equation in the standard form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

($a_{n-1}, \ldots a_1, a_0, f$ are all functions of x, and the DE above means that equality holds for all value of x in the interval I.)

This DE is called linear because the operator L defined by

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y$$

satisfies the so-called linearity properties

(1) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(2) $L(c\,y) = c\,L(y),\ c \in \mathbb{R}$.

• *The proof that L satisfies the linearity proporties is just the same as it was for the case when n = 2, that we checked Wednesday. Then, since the $y = y_P + y_H$ proof only depended on the linearity properties of L, just like yesterday, we deduce both of Theorems 0 and 1:*

Theorem 0: The solution space to the homogeneous linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

is a subspace.

Theorem 1: The general solution to the nonhomogeneous n^{th} order linear DE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

is $y = y_P + y_H$ where y_P is any single particular solution and y_H is the general solution to the homogeneous DE. (y_H is called y_c, for complementary solution, in the text).

Later in the course we'll understand n^{th} order existence uniqueness theorems for initial value problems, in a way analogous to how we understood the first order theorem using slope fields, but let's postpone that discussion and just record the following true theorem as a fact:

Theorem 2 (Existence-Uniqueness Theorem): Let $a_{n-1}(x), a_{n-2}(x), \ldots a_1(x), a_0(x), f(x)$ be specified continuous functions on the interval I, and let $x_0 \in I$. Then there is a unique solution $y(x)$ to the initial value problem

$$
\begin{aligned}
y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y &= f \\
y(x_0) &= b_0 \\
y'(x_0) &= b_1 \\
y''(x_0) &= b_2 \\
&\vdots \\
y^{(n-1)}(x_0) &= b_{n-1}
\end{aligned}
$$

and $y(x)$ exists and is n times continuously differentiable on the entire interval I.

Just as for the case n = 2, the existence-uniqueness theorem lets you figure out the dimension of the solution space to homogeneous linear differential equations. The proof is conceptually the same, but messier to write down because the vectors and matrices are bigger.

Theorem 3: The solution space to the n^{th} order homogeneous linear differential equation

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y \equiv 0$$

is n-dimensional. Thus, any n independent solutions $y_1, y_2, \ldots y_n$ will be a basis, and all homogeneous solutions will be uniquely expressible as linear combinations

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

proof: By the existence half of Theorem 2, we know that there are solutions for each possible initial value problem for this (homogenenous case) of the IVP for n^{th} order linear DEs. So, pick solutions $y_1(x), y_2(x), \ldots. y_n(x)$ so that their vectors of initial values (which we'll call initial value vectors)

$$
\begin{bmatrix} y_1(x_0) \\ y_1'(x_0) \\ y_1''(x_0) \\ \vdots \\ y_1^{(n-1)}(x_0) \end{bmatrix},
\begin{bmatrix} y_2(x_0) \\ y_2'(x_0) \\ y_2''(x_0) \\ \vdots \\ y_2^{(n-1)}(x_0) \end{bmatrix}, \ldots,
\begin{bmatrix} y_n(x_0) \\ y_n'(x_0) \\ y_n''(x_0) \\ \vdots \\ y_n^{(n-1)}(x_0) \end{bmatrix}
$$

are a basis for $\mathbb{R}^n$ (i.e. these n vectors are linearly independent and span $\mathbb{R}^n$. (Well, you may not know how to "pick" such solutions, but you know they exist because of the existence theorem.)

Claim: In this case, the solutions $y_1, y_2, \ldots y_n$ are a basis for the solution space. In particular, every solution to the homogeneous DE is a unique linear combination of these n functions and the dimension of the solution space is n .... discussion on next page.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the Wronskian matrix at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y(x) = z(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Thus $y_1, y_2, \ldots y_n$ span the solution space.

- **linear independence**: If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then differentiate this identity $n-1$ times, and then substitute $x = x_0$ into the resulting n equations. This yields the Wronskian matrix equation above, with $\left[b_0, b_1, \ldots b_{n-1}\right]^T = [0, 0, \ldots, 0]^T$. So the matrix equation above implies that $\left[c_1, c_2, \ldots c_n\right]^T = \mathbf{0}$. So $y_1, y_2, \ldots y_n$ are also linearly independent.

- Thus $y_1, y_2, \ldots y_n$ are a basis for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

□

Let's do some new exercises that tie these ideas together. (We may do these excersises while or before we wade through the general discussions on the previous pages!

Exercise 1) Consider the 3^{rd} order linear homogeneous DE for $y(x)$:

$$y''' + 3y'' - y' - 3y = 0.$$

Find a basis for the 3-dimensional solution space, and the general solution. Make sure to use the Wronskian matrix (or determinant) to verify you have a basis. Hint: try exponential functions.

try $e^{rx} = y$. $\Rightarrow y''' + 3y'' - y' - 3y = r^3 e^{rx} + 3r^2 e^{rx} - r e^{rx} - 3e^{rx}$
$= e^{rx}(r^3 + 3r^2 - r - 3) = 0$ for all x (want)

$p(r)$ "characteristic polynomial" — we want its roots

$r^3 + 3r^2 - r - 3$
$= r^2(r+3) - (r+3)$
$= (r^2-1)(r+3)$
$= (r-1)(r+1)(r+3) = 0$

roots $r = 1, -1, -3$.

soltns $y_1(x) = e^x$, $y_2(x) = e^{-x}$, $y_3(x) = e^{-3x}$

$$W = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} e^x & e^{-x} & e^{-3x} \\ e^x & -e^{-x} & -3e^{-3x} \\ e^x & e^{-x} & 9e^{-3x} \end{bmatrix}$$

@ $x=0$: $Wron = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 8 \end{vmatrix} \begin{matrix} \\ -R_1+R_2 \\ -R_1+R_3 \end{matrix} = -16 \neq 0!$$

Exercise 2a) Find the general solution to

$$y''' + 3y'' - y' - 3y = 6.$$

Hint: First try to find a particular solution ... try a constant function.

$y = y_P + y_H$

$y_P = -2$ works!

$y_P = d$ cnsts
$L(y_P) = 0 + 3\cdot 0 - 0 - 3d = 6$
$d = -2$ works.

$y = -2 + c_1 e^x + c_2 e^{-x} + c_3 e^{-3x}$
$y' = 0 + c_1 e^x - c_2 e^{-x} - 3c_3 e^{-3x}$
$y'' = 0 + c_1 e^x + c_2 e^{-x} + 9c_3 e^{-3x}$

b) Set up the linear system to solve the initial value problem for this DE, with $y(0) = -1, y'(0) = 2, y''(0) = 7$.

@ $x=0$:
$$\begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 9 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

↑ there's that Wronskian matrix @ $x=0$, again

for fun now, but maybe not just for fun later:

```
> with(DEtools) :
  dsolve({y'''(x) + 3·y''(x) − y'(x) − 3·y(x) = 6, y(0) = −1, y'(0) = 2, y''(0) = 7});
```

$$y(x) = -2 + \frac{9}{4}e^x + \frac{3}{4}e^{-3x} - 2e^{-x} \quad (1)$$

Math 2250-004
Week 9: March 6-10 5.2-5.4

Mon Mar 6: Section 5.2 from last week.

The two main goals in Chapter 5 are to learn the structure of solution sets to n^{th} order linear DE's, including how to solve the IVPs

$$\text{IVP}\left\{\begin{aligned} y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y &= f \\ y(x_0) &= b_0 \\ y'(x_0) &= b_1 \\ y''(x_0) &= b_2 \\ &\vdots \\ y^{(n-1)}(x_0) &= b_{n-1} \end{aligned}\right.$$

$y' + p(x)y = q(x)$
Chptr 1.

and to learn important physics/engineering applications of these general techniques.

Finish Wednesday March 1 notes from last week which discuss the case $n = 2$, and continue into the Friday March 3 notes which discuss the analogous ideas for general n. This is section 5.2 of the text.

Hw: 5.2.16 $\quad y''' - 5y'' + 8y' - 4y = 0$

$$\left.\begin{aligned} y(0) &= 1 \\ y'(0) &= 4 \\ y''(0) &= 0 \end{aligned}\right\}$$

$y_1 = e^x$
$y_2 = e^{2x}$
$y_3 = xe^{2x}$

$y(x) = c_1 y_1 + c_2 y_2 + c_3 y_3$
$y(x) = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}$
$y'(x) = c_1 e^x + 2c_2 e^{2x} + c_3 (e^{2x} + 2xe^{2x})$
$y''(x) = c_1 e^x + 4c_2 e^{2x} + c_3 (2e^{2x} + 2e^{2x} + 4xe^{2x})$

@ $x=0$: $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ Solve!

Tues Mar 7 (covering this on Wednesday)

5.3: Algorithms for the basis and general (homogeneous) solution to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

when the coefficients $a_{n-1}, a_{n-2}, \ldots a_1, a_0$ are all constant.

step 1) Try to find a basis made of exponential functions....try $y(x) = e^{r x}$. In this case

$$L(y) = e^{r x}\left(r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r x} p(r).$$

We call this polynomial $p(r)$ the characteristic polynomial for the differential equation, and can read off what it is directly from the expression for $L(y)$. For each root r_j of $p(r)$, we get a solution $e^{r_j x}$ to the homogeneous DE.

$p(r) = (r - r_1)(r - r_2) \cdots (r - r_n)$

Case 1) If $p(r)$ has n distinct (i.e. different) real roots $r_1, r_2, \ldots, r_n$, then

$$e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$$

is a basis for the solution space; i.e. the general solution is given by

$$y_H(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots + c_n e^{r_n x}.$$

Example: Exercise 1 from last Friday's notes: The differential equation (last example Tuesday)

$$y''' + 3y'' - y' - 3y = 0$$

has characteristic polynomial

$$p(r) = r^3 + 3r^2 - r - 3 = (r+3)\cdot(r^2 - 1) = (r+3)(r+1)\cdot(r-1) = 0$$

so the general solution to (solns e^{-3x}, e^{-x}, e^{x})

$$y''' + 3y'' - y' - 3y = 0$$

is

$$y_H(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{-3x}.$$

Exercise 1) By construction, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ all solve the differential equation. Show that they're linearly independent. This will be enough to verify that they're a basis for the solution space, since we know the solution space is n-dimensional. Hint: The easiest way to show this is to list your roots so that $r_1 < r_2 < \ldots < r_n$ and to use a limiting argument.

$\{e^{-2x}, e^{-x}, 1, e^{3x}\}$ is linearly independent (hence basis for order 4 L)

e.g. $p(r) = (r+2)(r+1)(r-0)(r-3)$

$c_1 e^{-2x} + c_2 e^{-x} + c_3 + c_4 e^{3x} = 0$ for all x, in $\mathbb{R}$.

$\div e^{3x}$: $c_1 e^{-5x} + c_2 e^{-4x} + c_3 e^{-3x} + c_4 = 0 \quad \forall x$

$\lim_{x \to \infty}$: $0 + 0 + 0 + \boxed{c_4 = 0}$

$\Rightarrow c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0 \quad \forall x$

$\lim_{x \to \infty}$: $\boxed{c_3 = 0}$

$\Rightarrow c_1 e^{-2x} + c_2 e^{-x} = 0$

$\div e^{-x}$: $c_1 e^{-x} + c_2 = 0$; $\lim_{x \to \infty}$ implies $\boxed{c_2 = 0} \Rightarrow \boxed{c_1 = 0}$

Case 2) Repeated real roots. In this case $p(r)$ has all real roots $r_1, r_2, \ldots r_m$ $(m < n)$ with the r_j all different, but some of the factors $(r - r_j)$ in $p(r)$ appear with powers bigger than 1. In other words, $p(r)$ factors as

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2}\cdots(r - r_m)^{k_m}$$

with some of the $k_j > 1$, and $k_1 + k_2 + \ldots + k_m = n$.

Start with a small example: The case of a second order DE for which the characteristic polynomial has a double root.

Exercise 2) Let r_1 be any real number. Consider the homogeneous DE

$$L(y) := y'' - 2\,r_1\,y' + r_1^2\,y = 0.$$

with $p(r) = r^2 - 2\,r_1\,r + r_1^2 = (r - r_1)^2$, i.e. r_1 is a double root for $p(r)$. Show that $e^{r_1 x}, x\,e^{r_1 x}$ are a basis for the solution space to $L(y) = 0$, so the general homogeneous solution is

$y_H(x) = c_1 e^{r_1 x} + c_2\,x\,e^{r_1 x}$. Start by checking that $x\,e^{r_1 x}$ actually (magically?) solves the DE.

(We may wish to study a special case $y'' + 6\,y' + 9\,y = 0$.)

$L(y) =$

$p(r) = r^2 + 6r + 9$
$= (r+3)^2 = 0$ $r = -3$, double root

$y_1(x) = e^{-3x}$, $y_2(x) = xe^{-3x}$

$L(e^{-3x})$
$= 9e^{-3x} + 6(-3e^{-3x}) + 9e^{-3x}$
$= (9 - 18 + 9)e^{-3x}$
$= 0$ ✓

$9[y_2 = xe^{-3x}]$
$+ 6[y_2' = e^{-3x} - 3xe^{-3x}]$
$+ 1[y_2'' = -3e^{-3x} - 3e^{-3x} + 9xe^{-3x} = -6e^{-3x} + 9xe^{-3x}]$

$L(y_2) = xe^{-3x}(9 - 18 + 9) + e^{-3x}(6 - 6)$
$= 0\,xe^{-3x} + 0e^{-3x}$
$= 0$

indep

$c_1 e^{-3x} + c_2 xe^{-3x} = 0$ for all x

$\div e^{-3x} \Rightarrow c_1 + c_2 x = 0$ for all x

@ $x = 0 \Rightarrow c_1 = 0$
@ $x = 1 \Rightarrow c_2 = 0$

Here's the general algorithm: If

$$p(r) = (r - r_1)^{k_1}(r - r_2)^{k_2} \dots (r - r_m)^{k_m}$$

then (as before) $e^{r_1 x}, e^{r_2 x}, \dots, e^{r_m x}$ are independent solutions, but since $m < n$ there aren't enough of them to be a basis. Here's how you get the rest: For each $k_j > 1$, you actually get independent solutions

$$\{ e^{r_j x}, x e^{r_j x}, x^2 e^{r_j x}, \dots, x^{k_j - 1} e^{r_j x} \}.$$

This yields k_j solutions for each root r_j, so since $k_1 + k_2 + \dots + k_m = n$ you get a total of n solutions to the differential equation. There's a good explanation in the text as to why these additional functions actually do solve the differential equation, see pages 316-318 and the discussion of "polynomial differential operators". I've also made a homework problem in which you can explore these ideas. Using the limiting method we discussed earlier, it's not too hard to show that all n of these solutions are indeed linearly independent, so they are in fact a basis for the solution space to $L(y) = 0$.

Exercise 3) Explicitly antidifferentiate to show that the solution space to the differential equation for $y(x)$

$$L(y) = y^{(4)} - y^{(3)} = 0$$

agrees with what you would get using the repeated roots algorithm in Case 2 above. Hint: first find $v = y'''$, using $v' - v = 0$, then antidifferentiate three times to find y_H. When you compare to the repeated roots algorithm, note that it includes the possibility $r = 0$ and that $e^{0x} = 1$.

recipe $L(e^{rx}) = e^{rx}(r^4 - r^3)$, $p(r) = r^4 - r^3$

$r^4 - r^3 = r^3(r-1)$
$= (r-0)^3(r-1)$

$y_1(x) = e^x$
$y_2(x) = 1$ e^{0x}
$y_3(x) = x$ xe^{0x}
$y_4(x) = x^2$ x^2e^{0x}

Chptr 1: $v = y^{(3)}(x)$

$v' - v = 0$
$e^{-x}(v' - v) = 0$
$\frac{d}{dx}(e^{-x}v) = 0$
$e^{-x}v = C$
$v = Ce^x$
$y''' = Ce^x$
$y'' = Ce^x + D$
$y' = Ce^x + Dx + E$
$y = Ce^x + \frac{1}{2}Dx^2 + Ex + F$
$= c_1e^x + c_4x^2 + c_3x + c_2$

Case 3) Complex number roots - this will be our surprising and fun topic on Wednesday. Our analysis will explain exactly how and why trig functions and mixed exponential-trig-polynomial functions show up as solutions for some of the homogeneous DE's you worked with in your homework for this week. This analysis depends on Euler's formula, one of the most beautiful and useful formulas in mathematics:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$
$$\text{for } i^2 = -1.$$

Wed 3/8: Use Tuesday notes & introduce Wed notes
• quiz at end of class

Friday: start here

Math 2250-004
Wed Mar 8

5.3 continued. How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work. $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \dots + a_1 y' + a_0 y = 0$

Strategy: In all cases we first try to find a basis for the n-dimensional solution space made of or related to exponential functions....trying $y(x) = e^{r x}$ yields

$$L(y) = e^{r x}\left(r^n + a_{n-1}r^{n-1} + \dots + a_1 r + a_0\right) = e^{r x}p(r) \; . = 0$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. There are three cases, of which the first two (distinct and repeated real roots) are in yesterday's notes.

Case 3) $p(r)$ has complex number roots. This is the hardest, but also most interesting case.The punch line is that exponential functions $e^{r x}$ still work, except that $r = a \pm b\,i$; but, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we need to learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions actually are important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \dots + \frac{1}{n!}f^{(n)}(0)x^n + \dots$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also needed error estimates to figure out on which intervals the Taylor polynomials actually coverged back to f.)

$f(x) \approx p_n(x)$ so that $f(0) = p(0)$, $f'(0) = p'(0)$ ... $f^{(n)}(0) = p^{(n)}(0)$.

$p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$
$f(0) = p(0) = a_0$ $\quad p'(x) = a_1 + 2a_2 x + 3a x^2 + \dots$
$f'(0) = p'(0) = a_1$
$f''(0) = p''(0) = 2a_2 \Rightarrow a_2 = \frac{1}{2}f''(0)$
$f'''(0) = p'''(0) = 6a \Rightarrow a_3 = \frac{1}{3!}f'''(0)$

Exercise 1) Use the formula above to recall the three very important Taylor series for

1a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$
1b) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
1c) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

In Calculus you checked that these series actually converge and equal the given functions, for all real numbers x.

$f(x) = e^x$, $f^{(n)}(x) = e^x$ | @ $x=0$: 1

$g(x) = \cos x$, $g' = -\sin x$, $g'' = -\cos x$, $g''' = \sin x$, $g^{(4)} = \cos x$ | @ $x=0$: 1, 0, −1, 0

$h(x) = \sin x$, $h' = \cos x$, $h'' = -\sin x$, $h''' = -\cos x$, $h^{(4)} = \sin x$ | @ $x=0$: 0, 1, 0, −1

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$e^{i\,\theta} = \cos(\theta) + i\sin(\theta) \; .$$

$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta}$ HW

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \dots$$

$\cos\theta$ $\quad i\sin\theta$

From Euler's formula it makes sense to define

$$e^{a+bi} := e^a e^{bi} = e^a(\cos(b) + i\sin(b))$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$$e^{(a+bi)x} = e^{ax}(\cos(bx) + i\sin(bx)) = e^{ax}\cos(bx) + i\, e^{ax}\sin(bx)\,.$$

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x(f(x) + i\,g(x)) := f'(x) + i\,g'(x)\,.$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

Exercise 3) Check that $D_x\left(e^{(a+bi)x}\right) = (a+bi)e^{(a+bi)x}$, i.e.

$$D_x\, e^{rx} = r\, e^{rx}$$

even if r is complex. (So also $D_x^2\, e^{rx} = D_x r\, e^{rx} = r^2\, e^{rx}$, $D_x^3\, e^{rx} = r^3\, e^{rx}$, etc.)

Now return to our differential equation questions, with

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y\,.$$

Then even for complex $r = a + bi$ $(a, b \in \mathbb{R})$, our work above shows that

$$L\left(e^{rx}\right) = e^{rx}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{rx}p(r)\,.$$

So if $r = a + bi$ is a complex root of $p(r)$ then e^{rx} is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that

$$0 + 0\,i = L\left(e^{rx}\right) = L\left(e^{ax}\cos(bx) + i\,e^{ax}\sin(bx)\right)$$
$$= L\left(e^{ax}\cos(bx)\right) + i\,L\left(e^{ax}\sin(bx)\right).$$

Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce

$$0 = L\left(e^{ax}\cos(bx)\right)$$
$$0 = L\left(e^{ax}\sin(bx)\right).$$

Upshot: If $r = a + bi$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{ax}\cos(bx)$$
$$y_2 = e^{ax}\sin(bx)$$

are two solutions to $L(y) = 0$. (The conjugate root $a - bi$ would give rise to $y_1, -y_2$, which have the same span.

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a x}\cos(b x),\; e^{a x}\sin(b x) \\ x\,e^{a x}\cos(b x),\; x\,e^{a x}\sin(b x) \\ \vdots \qquad\qquad \vdots \\ x^{k-1}e^{a x}\cos(b x),\; x^{k-1}e^{a x}\sin(b x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 4) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 4\,y = 0\,.$$

(You were told a basis in the last problem of last week's hw....now you know where it came from.)

$p(r) = r^2 + 4 = 0$
$r^2 = -4$
$r = \pm 2i$ $\quad a = 0$, $b = 2$

recipe: $y_1(x) = \cos 2x$
$y_2(x) = \sin 2x$

$r = a \pm ib$
$y_1(x) = e^{ax}\cos bx$
$y_2(x) = e^{ax}\sin bx$

Exercise 5) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0\,.$$

$p(r) = r^2 + 6r + 13 = 0$
$(r+3)^2 + 4 = 0$
$(r+3)^2 = -4$
$r + 3 = \pm 2i$
$r = -3 \pm 2i$ $\quad a = -3$, $b = 2$

$y_1(x) = e^{-3x}\cos 2x$
$y_2(x) = e^{-3x}\sin 2x$

Exercise 6) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = \left(r^2 + 6\,r + 13\right)^2 (r - 2)^3\,.$$

What is the general solution to the corresponding homogeneous DE?

$\left((r+3)^2 + 4\right)^2 (r-2)^3$ $\quad [4 = -(2i)^2]$
$= \left((r+3+2i)(r+3-2i)\right)^2 (r-2)^3$
$= (r+3+2i)^2 (r+3-2i)^2 (r-2)^3$

$\mathbb{C}$ roots $-3 \mp 2i$

$y_H(x) = c_1 e^{-3x}\cos 2x + c_2 e^{-3x}\sin 2x + c_3 x e^{-3x}\cos 2x + c_4 x e^{-3x}\sin 2x$
$+ c_5 e^{2x} + c_6 x e^{2x} + c_7 x^2 e^{2x}$

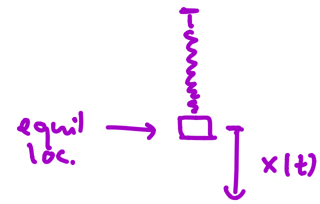
Friday : Wednesday notes
at end of class, introduce § 5.4

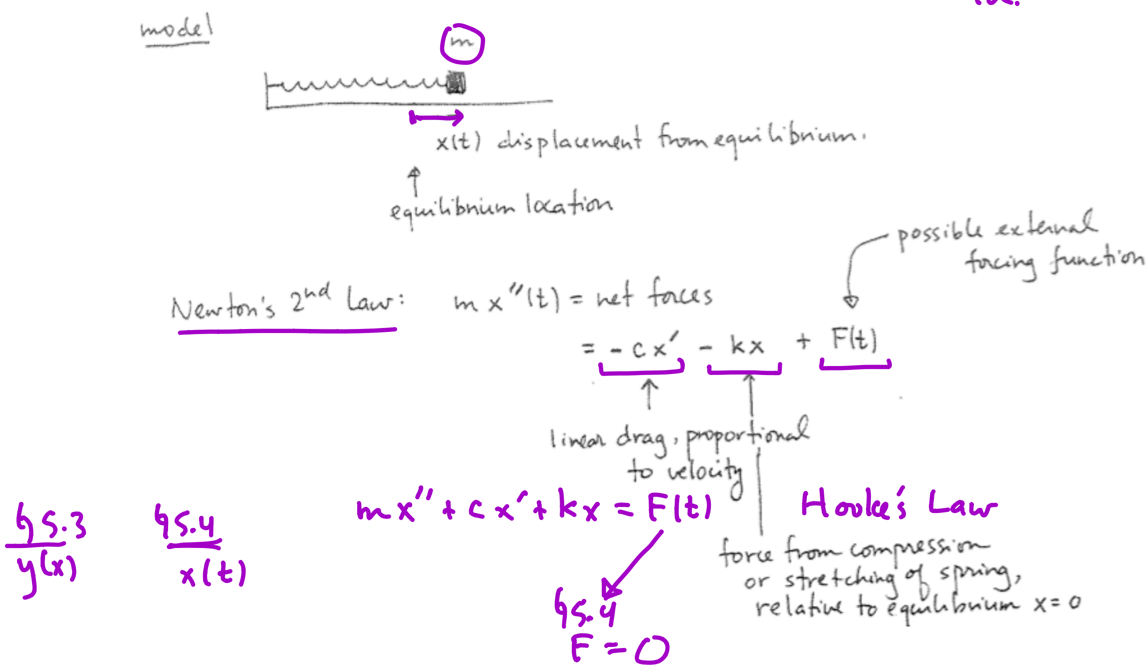
Friday Mar 10

5.4: Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$:

$$m\,x'' + c\,x' + k\,x = 0\,.$$





In section 5.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero. Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t)$, $x'(t)$ are sufficiently small. k is called the <u>Hooke's constant</u>, and c is called the <u>damping coefficient</u>.

$mx'' + cx' + kx = 0$

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{rt}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{rt}\left(m\,r^2 + c\,r + k\right) = e^{rt}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ processes...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}\,x = 0\,.$$
$$p(r) = r^2 + \frac{k}{m}\,,$$

has roots

$$r^2 = -\frac{k}{m} \qquad \text{i.e.} \qquad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

So the general solution is

$$x(t) = c_1\cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the natural angular frequency. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\,x = 0$$

is

$$x(t) = A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right).$$

This motion is called simple harmonic motion. The reason for this is that $x(t)$ can be rewritten as

$$x(t) = C\cos\left(\omega_0\,t - \alpha\right) = C\cos\left(\omega_0\left(t - \delta\right)\right)$$

in terms of an amplitude $C > 0$ and a phase angle α (or in terms of a time delay δ).

To see why functions of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

are equal (for appropriate choices of constants) to ones of the form

$$x(t) = C\cos(\omega_0 t - \alpha)$$

we use the very important the addition angle trigonometry identities, in this case the addition angle for *cosine*: Consider the possible equality of functions

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha).$$

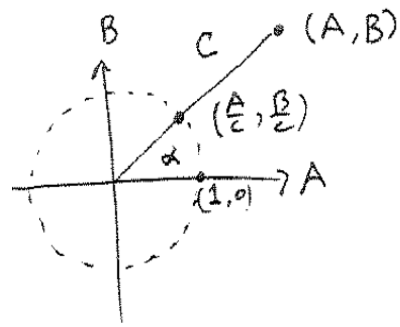
Exercise 1) Use the addition angle formula $\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ to show that the two functions above are equal provided

$$A = C\cos\alpha$$
$$B = C\sin\alpha\ .$$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:



It is important to understand the behavior of the functions

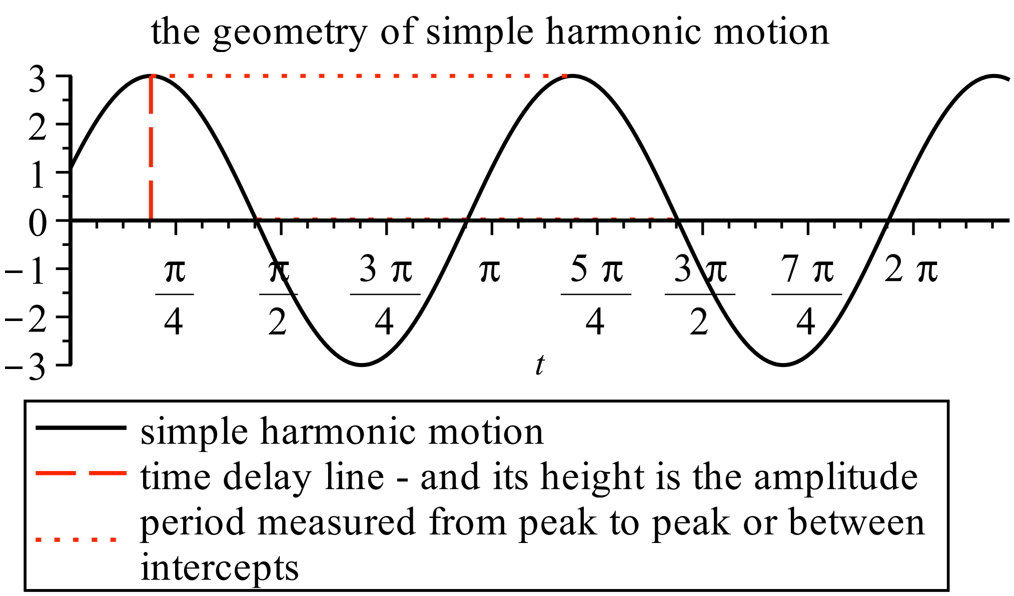
$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} = \text{time/cycle.}$$



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0 ..7, color = black) :
  plot2 := plot([.6, t, t = 0 ..3.], linestyle = dash) :
  plot3 := plot(3, t = .6 ..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

Exercise 2) A mass of 2 *kg* oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 *m* from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

2a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$x'' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

2b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the commands below.

```
> unassign(`x`);
> with(plots):
  with(DEtools):
> dsolve({x''(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
> plot(rhs(%), t = 0..5, color = green);
>
```

• Then, if time, discuss the possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

Case 2: damping

$$m\,x'' + c\,x' + k\,x = 0$$
$$x'' + \frac{c}{m}\,x' + \frac{k}{m}\,x = 0$$

rewrite as

$$x'' + 2\,p\,x' + \omega_0^2\,x = 0.$$

$\left(p = \frac{c}{2\,m},\ \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$r^2 + 2\,p\,r + \omega_0^2 = 0$$

which has roots

$$r = \frac{-2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = -p \pm \sqrt{p^2 - \omega_0^2}\,.$$

2a) ($p^2 > \omega_0^2$, or $c^2 > 4\,m\,k$). overdamped. In this case we have two negative real roots

$$r_1 < r_2 < 0$$

and

$$x(t) = c_1 e^{r_1 t} + c_2\,e^{r_2 t} = e^{r_2 t}\left(c_1 e^{(r_1 - r_2)\,t} + c_2\right).$$

• solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

2b) ($p^2 = \omega_0^2$, or $c^2 = 4\,m\,k$) critically damped. Double real root $r_1 = r_2 = -p = -\frac{c}{2\,m}$.

$$x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right).$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

2c) ($p^2 < \omega_0^2$, or $c^2 < 4\,m\,k$) underdamped . Complex roots

$$r = -p \pm \sqrt{p^2 - \omega_0^2} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$$x(t) = e^{-p\,t}\left(A\cos\left(\omega_1 t\right) + B\sin\left(\omega_1\,t\right)\right) = e^{-p\,t}C\cos\left(\omega_1 t - \alpha_1\right).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1 .

Exercise 3) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:

3a)

$$x'' + 6\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

```
> with(DEtools):
> dsolve({x''(t) + 6·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
>
```

3b)

$$x'' + 10\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

```
> dsolve({x''(t) + 10·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
>
```

3c)

$$x'' + 2\,x' + 9\,x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}\,.$$

```
> dsolve({x''(t) + 2·x'(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
>
```

```
> with(plots):
> plot0 := plot(cos(3·t) + 1/2·sin(3·t), t = 0..4, color = red):
  plot1a := plot(exp(-3·t)·(1 + 9/2·t), t = 0..4, color = green):
  plot1b := plot(21/16·exp(-t) - 5/16·exp(-9·t), t = 0..4, color = blue):
  plot1c := plot(5/8·sqrt(2)·e^(-t)·sin(2·sqrt(2)·t) + e^(-t)·cos(2·sqrt(2)·t), t = 0..4, color = black):
  display({plot0, plot1a, plot1b, plot1c}, title = `IVP with all damping possibilities`);
```