Exercise 4) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\omega^2 << \omega_0^2$ we have α near zero (in phase) for x_{sp}, because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have α near π (out of phase), because $\sin(\alpha) \approx 0$, $\cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1$, $\cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

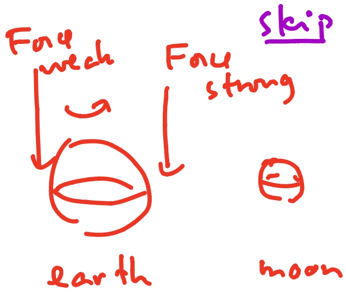
Exercise 5) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0.$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}\,e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3).$$

skip



2 high tides/day

forcing period 12 hours

ω_0 ? $T_0 \approx 2$ days

$\omega >> \omega_0$

expect tides to be 180° (π-rad) out of phase.

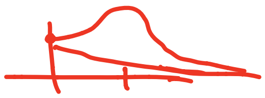
oops

> with(DEtools):

> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});

$$x(t) = -3\,e^{-t}\sin(5t) + e^{-t}\cos(5t) + 4\sin(4t) + 5\cos(4t) \quad \textbf{(14)}$$

Practical resonance: The steady periodic amplitude C for damped forced oscillations is



$$C(\omega) = \frac{F_0}{\sqrt{(k - m\,\omega^2)^2 + c^2\omega^2}}\,.$$

$C(0) = \dfrac{F_0}{k}.$

Notice that as $\omega \to 0$, $C(\omega) \to \dfrac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\dfrac{F_0}{k}$, on the interval $0 < \omega < \infty$.

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 6a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

6b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

$c = 0$

$\omega_0 = \sqrt{26}$

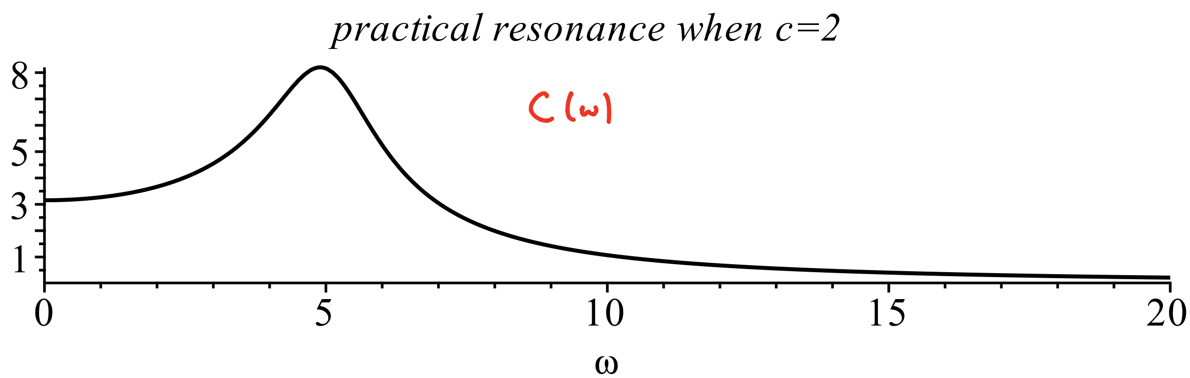
$$C(\omega) = \frac{82}{\sqrt{(26-\omega^2)^2 + c^2\omega^2}} = \frac{82}{\sqrt{(26-\omega^2)^2 + 4\omega^2}}$$

> *restart* :

> *with*(*plots*) :

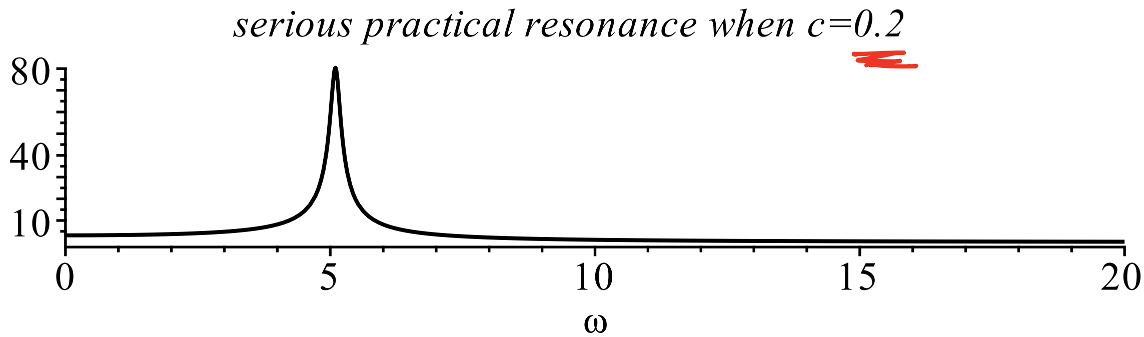
> $C := (\omega, c) \to \dfrac{82}{\sqrt{(26-\omega^2)^2 + c^2\cdot\omega^2}}$:

> *plot*($C(\omega, 2)$, $\omega = 0..20$, *color* = *black*, *title* = \`*practical resonance when c=2*\`);

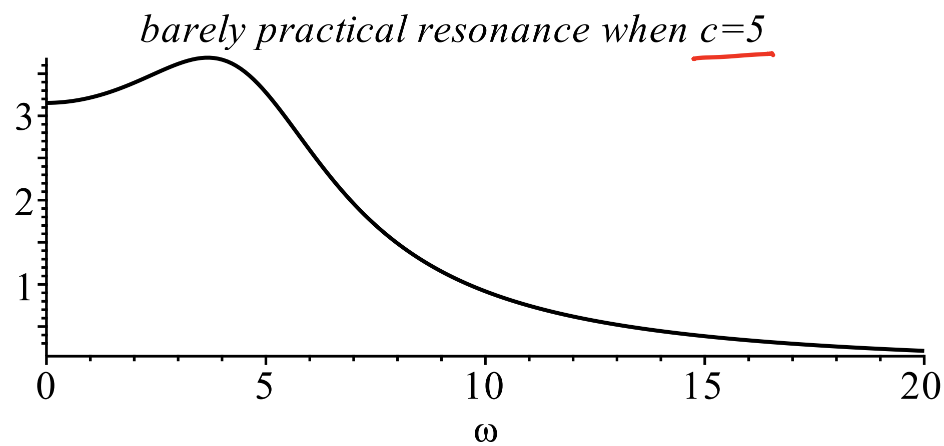


>

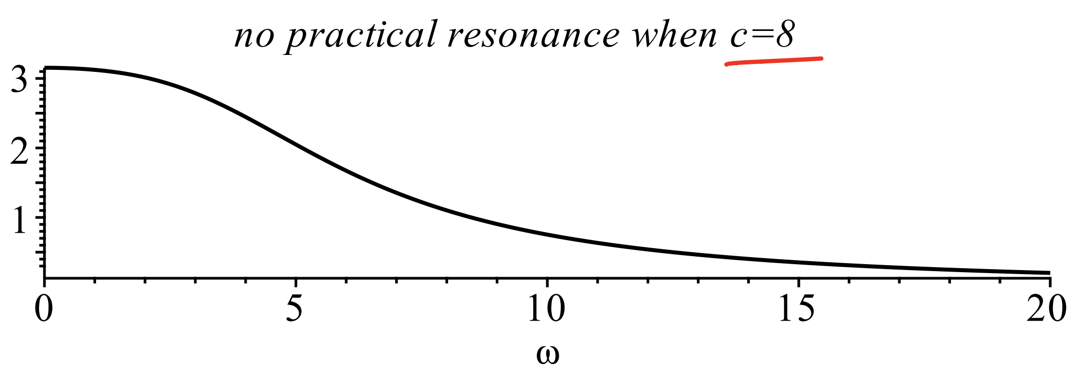
> plot($C(\omega, .2)$, $\omega = 0..20$, color = black, title = `serious practical resonance when c=0.2`);



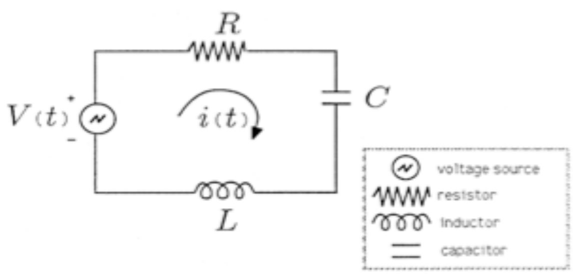
> plot($C(\omega, 5)$, $\omega = 0..20$, color = black, title = `barely practical resonance when c=5`);



> plot($C(\omega, 8)$, $\omega = 0..20$, color = black, title = `no practical resonance when c=8`);



<u>The mechanical-electrical analogy, continued:</u> Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....recall from earlier in the course:



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

<u>Kirchoff's Law</u>: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t) = \overset{120}{E_0}\sin(\overset{60}{\omega}\,t)$$ typical wall

$\frac{d}{dt}$

For $I(t)$: $$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t) = E_0\,\omega\cos(\omega\,t)\,.$$

$m x'' + c x' + k x = F_0 \cos\omega t$

Transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + \cancel{I_{tr}(t)}^{\to 0}\,.$$

$$I_{sp}(t) = I_0\cos(\omega\,t - \alpha) = I_0\sin(\omega\,t - \gamma)\,,\quad \gamma = \alpha - \frac{\pi}{2}\,.$$

$x_{sp}(t) = C(\omega)(\cos(\omega t - \alpha))$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \Rightarrow I_0(\omega) = \frac{E_0\cancel{\omega}}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow \boxed{I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}}}\,.$$

$= \cancel{\omega}\sqrt{\cancel{\omega^2}\left(\frac{1}{C\omega} - L\omega\right)^2 + R^2\cancel{\omega^2}}$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the <u>impedance</u> $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\,\omega} = L\,\omega$, i.e.

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old radios).}$$

$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable}$$

I_0 max when $\frac{1}{C\omega} = L\omega$, $\frac{1}{LC} = \omega^2$

Both L and C are adjusted in this M.I.T. lab demonstration: http://www.youtube.com/watch?v=ZYgFuUl9_Vs.

§5.6 forced oscillations.
Fri: no damping — beating $\omega \approx \omega_0$, $\omega \neq \omega_0$
— pure resonance $\omega = \omega_0$
Today Mon: damping — practical resonance
RLC analog

Math 2250-004 Week 11 March 27-31

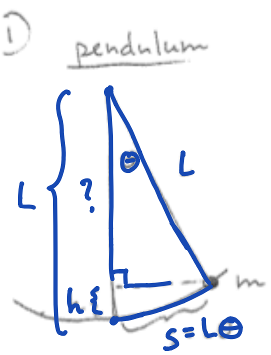
Mon Mar 27: Continue and maybe finish Friday's notes on section 5.6, forced oscillation phenomena

Tues Mar 28: Pendulum and mass-spring experiments from last Tuesday (notes included here, though). Then begin Laplace Transform, 10.1-10.2

Experiment discussion: Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \dfrac{2\pi}{\omega_0}$ correspond to experimental reality.

Today Tuesday
1) experiments
2) RLC circuits from old notes
→ I'll post 2 practice exams & review notes later today (on CANVAS)
Wed: Laplace Transform (not for this exam!)

1) pendulum



conservative system KE + PE = const.

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$

$h = L - L\cos\theta = L(1-\cos\theta)$

$\theta = \theta(t)$

$\frac{?}{L} = \cos\theta$

$? = L\cos\theta$

so, $\frac{1}{2}mL^2(\theta'(t))^2 + mgL(1-\cos\theta(t)) \equiv C$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$mL\theta'(\underbrace{L\theta'' + g\sin\theta}) \equiv 0$

$\neq 0$ except at isolated times

⇝ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

linearize

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$\omega_0 = \sqrt{\frac{k}{m}}$

↘ non-linear DE

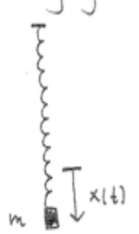
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$\theta < .1$
error if replace $\sin\theta$ by θ is at most $\frac{(.1)^3}{6} = \frac{.001}{6}$

2) hanging mass-spring:



$mx'' = -kx$

$mx'' + kx = 0$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

> *restart* :
> *Digits* := 4 :

> $L := 1.53;$ m
> $g := 9.806;$
> $\omega := \sqrt{\frac{g}{L}}$; # *radians per second*
> $f := evalf(\omega / (2 \cdot \text{Pi}))$; # *cycles per second*
> $T := 1/f$; # *seconds per cycle*

$$L := 1.53$$
$$g := 9.806$$
$$\omega := 2.531629974 \quad \omega \quad \text{rad/sec.}$$
$$f := 0.4029214244 \quad \text{cycles/sec.}$$
$$T := \underline{2.481}873486 \quad \text{sec/cycle} \qquad \textbf{(1)}$$

Experiment:

10 cycles : 24.3, 24.5, 24.55, 24.8, 24.65, 24.68, 24.4, 24.75, 24.67

$$\frac{24.6}{10} \frac{\text{sec}}{\text{cycle}} = \underline{\underline{2.46}}$$

Mass-spring:
compute Hooke's constant:

> $98.7 - 83.4$; #*displacement from extra 50g*

$$15.3 \quad \checkmark .156 \text{ m} \qquad \textbf{(2)}$$

> $k := \frac{.05 \cdot 9.806}{.153}$; # *solve k·x=m·g for k.*

$$k := 3.204575163 \quad \checkmark \quad k = 3.143 \qquad \textbf{(3)}$$

> $m := .1$; # *mass for experiment is 100g*
> $\omega := \sqrt{\frac{k}{m}}$; # *predicted angular frequency* $\quad \omega = \sqrt{\frac{3.143}{.1}} = .$
> $f := evalf\left(\frac{\omega}{2 \cdot \text{Pi}}\right)$; # *predicted frequency*
> $T := \frac{1}{f}$; # *predicted period*

$$m := 0.1$$
$$\omega := 5.660896716 \quad 5.606$$
$$f := 0.9009596945 \quad .892$$
$$T := 1.109927565 \quad \underline{1.12} \qquad \textbf{(4)}$$

Experiment:

20 cycles 23 sec, 22.97, 23.38, 23.17, 23.07, ~~23.27~~, 23.27, 23.5

$$\frac{23.19}{20} \frac{\text{sec}}{\text{cycle}} = \boxed{\underline{1.16} \text{ sec}}$$

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>
Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work} .$$

$$PE_{work} = \int_0^x k\, s\, \mathrm{d}s = \frac{1}{2} k\, x^2, \quad KE_{mass} = \frac{1}{2} m\, (x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\, x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2} (\mu\, x'(t))^2 \left(m_{spring}\, \mathrm{d}\mu\right)$$

$$= \frac{1}{2} m_{spring} (x'(t))^2 \int_0^1 \mu^2\, d\, \mu = \frac{1}{6} m_{spring} (x'(t))^2 .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3} m_{spring}\right)(x'(t))^2 + \frac{1}{2} k\, x^2 = \frac{1}{2} M (x'(t))^2 + \frac{1}{2} k\, x^2 ,$$

where

$$\boxed{M = m + \frac{1}{3} m_{spring}}$$

$\mathrm{D}_t(TE) = 0 \Rightarrow$

$$M\, x'(t)\, x''(t) + k\, x(t)\, x'(t) = 0 .$$
$$x'(t)\,(M\, x'' + k\, x) = 0 .$$

Since $x'(t) = 0$ only at isolated $t-$values, we deduce that the corrected equation of motion is

$$(M\, x'' + k\, x) = 0$$

with

$$\boxed{\omega_0 = \sqrt{\frac{k}{M}} .}$$

Does this lead to a better comparison between model and experiment?

```
> ms := .011;  # spring has mass 11g
  M := m + 1/3·ms;  # "effective mass"
```

$$\boxed{\begin{aligned} ms &:= 0.011 \\ M &:= 0.1036666667 \end{aligned}} \qquad (5)$$

$\sqrt{\frac{3.143}{.1037}}$

k = 3.143

> $\omega := \sqrt{\frac{k}{M}}$; *# predicted angular frequency*

$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right)$; *# predicted frequency*

$T := \frac{1}{f}$; *# predicted period*

$\omega := 5.559883146$ | 5.506

$f := 0.8848828855$ | .876

$T := 1.130093051$ | 1.14 sec/cycle **(6)**

= better

>