$$\bullet \quad x(t) = \frac{F_0}{2m\omega_0} t \sin\omega_0 t + x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

Exercise 3a) Solve the IVP

$$x'' + 9x = 80\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

Start here: $\quad \omega = 3 \qquad x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t$

$\omega_0 = \sqrt{9} = 3 \qquad \omega_0 = \sqrt{\frac{k}{m}}$

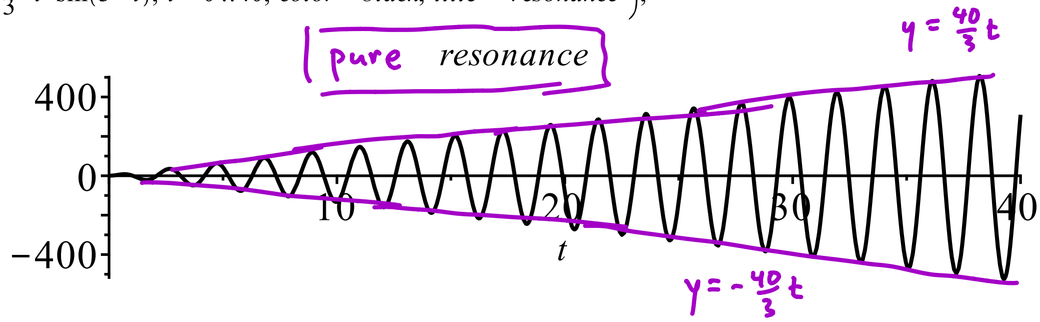
$\frac{F_0}{m} = 80$

$$x(t) = \underbrace{\frac{80}{6} t \sin 3t}_{\frac{40}{3} t \sin 3t} + 0 + 0$$

$T = \frac{2\pi}{3} \approx 2$

3b) Compare the solution graph below with the beating graph in exercise 2.

$> plot\left(\frac{40}{3} t\cdot\sin(3\cdot t),\ t = 0..40,\ color = black,\ title = \text{`resonance`}\right);$




>

• After finishing the discussion of undamped forced oscillations, we will discuss the physics and mathematics of damped forced oscillations

$$m x'' + c x' + k x = F_0 \cos(\omega t) .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

$$m x'' + c x' + k x = F_0 \cos\omega t \qquad \underline{c>0}$$

<u>guess:</u> $x_p = A\cos\omega t + B\sin\omega t$

$$\begin{aligned} k\,[\,x &= A\cos\omega t + B\sin\omega t\,] \\ c\,[\,x' &= -A\omega\sin\omega t + B\omega\cos\omega t\,] \\ m\,[\,x'' &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t\,] \end{aligned}$$

$$L(x) = \cos\omega t\,(kA + c\omega B - m\omega^2 A) + \sin\omega t\,(kB - c\omega A - m\omega^2 B) \overset{\text{want}}{=} \cos\omega t\,(F_0) + \sin\omega t\,(0)$$

$$\begin{bmatrix} k-m\omega^2 & c\omega \\ -c\omega & k-m\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} (k-m\omega^2)A + c\omega B &= F_0 \\ -c\omega A + (k-m\omega^2)B &= 0 \end{aligned}$$

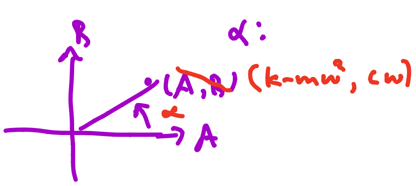
$M\vec{x} = \vec{b}$
$\vec{x} = M^{-1}\vec{b}$.

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + (c\omega)^2} \begin{bmatrix} k-m\omega^2 & -c\omega \\ c\omega & k-m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2} \begin{bmatrix} k-m\omega^2 \\ c\omega \end{bmatrix}$$

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \alpha)$$

$$C = \sqrt{A^2+B^2} = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\sqrt{(k-m\omega^2)^2 + c^2\omega^2} = \frac{F_0}{\sqrt{\underbrace{(k-m\omega^2)^2}_{\substack{\left(m\left(\frac{k}{m}-\omega^2\right)\right)^2 \\ (m^2)(\omega_0^2-\omega^2)^2}} + c^2\omega^2}}$$

$\alpha$:




$$\cos\alpha = \frac{k-m\omega^2}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}$$

$$\sin\alpha = \frac{c\omega}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}$$

Damped forced oscillations $(c > 0)$ for $x(t)$:

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k\,&\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\ +\,c\,&\left[x_P{}' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right] \\ +\,m\,&\left[x_P{}'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right] . \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega t)\left(k\,A + c\,B\,\omega - m\,A\,\omega^2\right) \\ &\quad + \sin(\omega t)\left(k\,B - c\,A\,\omega - m\,B\,\omega^2\right) . \end{aligned}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}\begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} . \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \longleftarrow m\left(\tfrac{k}{m} - \omega^2\right) = m\left(\omega_0^2 - \omega^2\right)$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} . \quad > 0$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)$ .
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p\,t}(c_1 t + c_2)$ .
- over-damped: $x = x_{sp} + x_{tr} = \underbrace{C\cos(\omega t - \alpha)}_{x_{sp}(t)} + \underbrace{c_1 e^{-r_1 t} + c_2 e^{-r_2 t}}_{x_H = x_{tr}(t)}$ .

Important to note:

- The amplitude $C$ in $x_{sp}$ can be quite large relative to $\frac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle $\alpha$ is always in the first or second quadrant.

Exercise 4) ( a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient $c$! Namely, for $c$ small, when $\omega^2 << \omega_0^2$ we have $\alpha$ near zero (in phase) for $x_{sp}$, because $\sin(\alpha) \approx 0, \cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have $\alpha$ near $\pi$ (out of phase), because $\sin(\alpha) \approx 0, \cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, $\alpha$ is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1, \cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

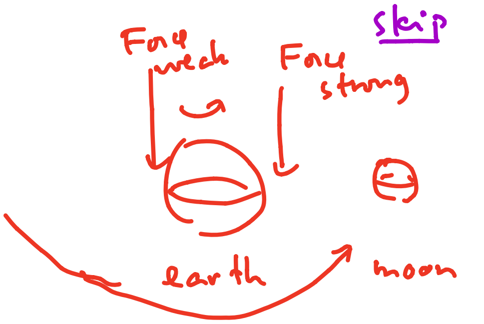
Exercise 5) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0.$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}\,e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8), \beta = \arctan(-3).$$

skip




2 high tides/day

forcing period 12 hours

$\omega_0$? $T_0 \approx 2$ days

$\omega >> \omega_0$

expect tides to be 180°

($\pi$-rad)

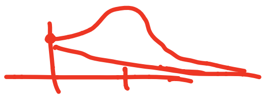
out of phase.

oops

```
> with(DEtools) :
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
```

$$x(t) = -3\,e^{-t}\sin(5t) + e^{-t}\cos(5t) + 4\sin(4t) + 5\cos(4t) \qquad \textbf{(14)}$$

Practical resonance: The steady periodic amplitude $C$ for damped forced oscillations is



$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} .$$

$C(0) = \frac{F_0}{k}.$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$ .

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in $\omega^2$ it will have at most one minimum in the variable $\omega^2$, so $C(\omega)$ will have at most one maximum for non-negative $\omega$. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 6a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient $c$:

$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t) .$$

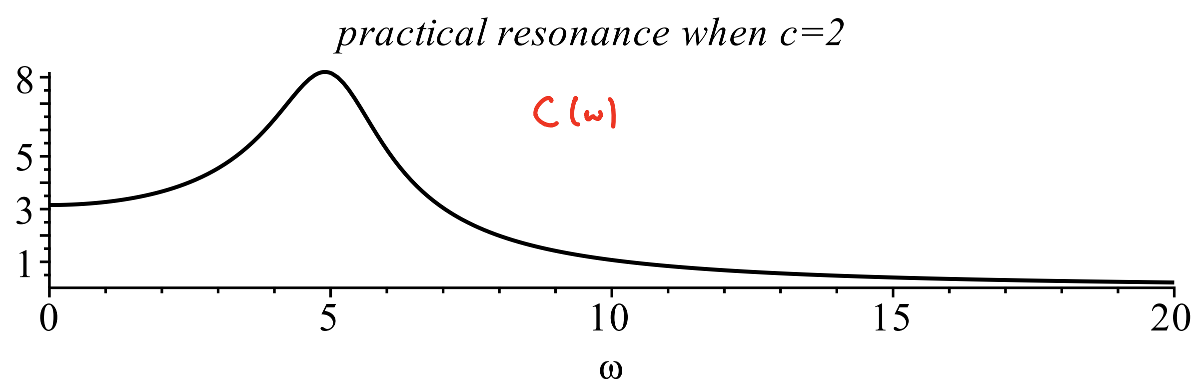
6b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

$c = 0$

$\omega_0 = \sqrt{26}$

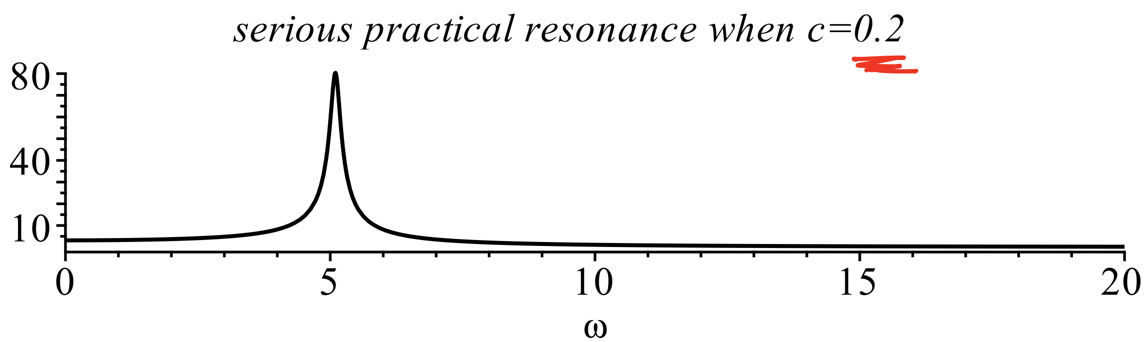
$$C(\omega) = \frac{82}{\sqrt{(26-\omega^2)^2 + c^2\omega^2}} = \frac{82}{\sqrt{(26-\omega^2)^2 + 4\omega^2}}$$

```
> restart :
> with(plots) :
> C := (ω, c) → 82 / sqrt((26 − ω^2)^2 + c^2·ω^2) :
> plot(C(ω, 2), ω = 0 ..20, color = black, title = `practical resonance when c=2`);
```

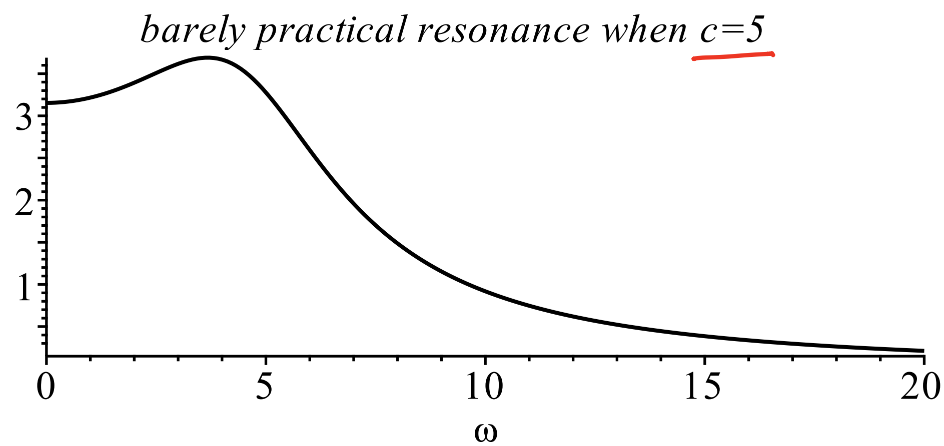



```
>
```

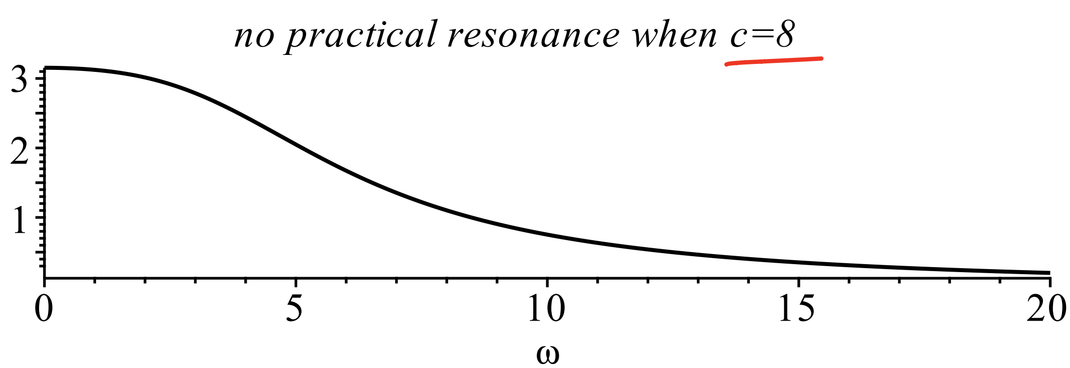
> plot( C( ω, .2 ), ω = 0 ..20, *color* = *black*, *title* = `*serious practical resonance when c=0.2*` );




> plot( C( ω, 5 ), ω = 0 ..20, *color* = *black*, *title* = `*barely practical resonance when c=5*` );




> plot( C( ω, 8 ), ω = 0 ..20, *color* = *black*, *title* = `*no practical resonance when c=8*` );



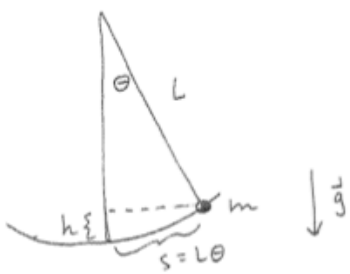
§5.6 forced oscillations.
Fri: no damping — beating $\omega \approx \omega_0,\ \omega \neq \omega_0$
— pure resonance $\omega = \omega_0$
Today Mon: damping — practical resonance
RLC analog

Math 2250-004 Week 11 March 27-31

Mon Mar 27: Continue and maybe finish Friday's notes on section 5.6, forced oscillation phenomena

Tues Mar 28: Pendulum and mass-spring experiments from last Tuesday (notes included here, though).
Then begin Laplace Transform, 10.1-10.2
Experiment discussion: Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \dfrac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum




$\theta = \theta(t)$

conservative system KE + PE = const.

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2}m\,L^2(\theta'(t))^2 + mgL\,(1-\cos(\theta(t))) \equiv \text{const}$

$D_t$: $mL^2\,\theta'\theta'' + mgL(\sin\theta)\,\theta' \equiv 0$

$\underbrace{mL\theta'}_{\neq 0 \text{ except at isolated times}}\,(L\theta'' + g\sin\theta) \equiv 0$

~ deduce eqtn of motion is

$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

linearize

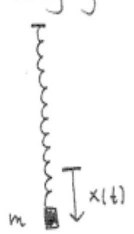
$\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

non-linear DE
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$
$\sin\theta \approx \theta$ $\quad\theta$ small
is excellent approx
(alternating series test)

2) hanging mass-spring:




$mx'' = -kx$

$mx'' + kx = 0$

$\boxed{x'' + \frac{k}{m}x = 0}$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?