Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2 (\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad) \\
0 (\quad & x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad) \\
+1 (\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$, $B = \frac{F_0}{2m\omega_0}$

$$\underbrace{\boxed{x_p(t) = \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t}}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\,t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class.

- $x(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$

Exercise 3a) Solve the IVP

$$x'' + 9x = 80\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

Start here: $\omega = 3$ $\qquad x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t$

$\omega_0 = \sqrt{9} = 3$ $\qquad \omega_0 = \sqrt{\frac{k}{m}}$

$\frac{F_0}{m} = 80$

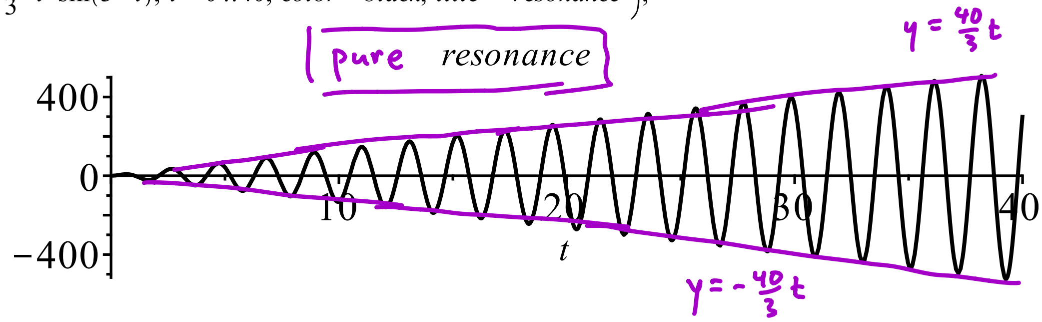
$$x(t) = \underbrace{\frac{80}{6} t \sin 3t}_{} + 0 + 0$$

$$\frac{40}{3} t \underbrace{\sin 3t}_{}$$

$$T = \frac{2\pi}{3} \approx 2$$

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t \cdot \sin(3 \cdot t),\ t = 0..40,\ color = black,\ title = \text{`resonance`}\right);$



>

• After finishing the discussion of undamped forced oscillations, we will discuss the physics and mathematics of damped forced oscillations

$$m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)\ .$$

Here are some links which address how these phenomena arise, also in more complicated real-world applications in which the dynamical systems are more complex and have more components. Our baseline cases are the starting points for understanding these more complicated systems. We'll also address some of these more complicated applications when we move on to systems of differential equations, in a few weeks.

http://en.wikipedia.org/wiki/Mechanical_resonance (wikipedia page with links)
http://www.nset.org.np/nset/php/pubaware_shaketable.php (shake tables for earthquake modeling)
http://www.youtube.com/watch?v=M_x2jOKAhZM (an engineering class demo shake table)
http://www.youtube.com/watch?v=j-zczJXSxnw (Tacoma narrows bridge)
http://en.wikipedia.org/wiki/Electrical_resonance (wikipedia page with links)
http://en.wikipedia.org/wiki/Crystal_oscillator (crystal oscillators)

$$mx'' + cx' + kx = F_0\cos\omega t \qquad \underline{c>0}$$

<u>guess</u>: $x_p = A\cos\omega t + B\sin\omega t$

$$\begin{aligned}
k\,[\,x &= A\cos\omega t + B\sin\omega t\,]\\
c\,[\,x' &= -A\omega\sin\omega t + B\omega\cos\omega t\,]\\
m\,[\,x'' &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t\,]
\end{aligned}$$

$$L(x) = \cos\omega t\,(kA + c\omega B - m\omega^2 A) + \sin\omega t\,(kB - c\omega A - m\omega^2 B) \overset{\text{want}}{=} \cos\omega t\,(F_0) + \sin\omega t\,(0)$$

$$\begin{bmatrix} k-m\omega^2 & c\omega \\ -c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B\end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$(k-m\omega^2)A + c\omega B = F_0$$
$$-c\omega A + (k-m\omega^2)B = 0$$

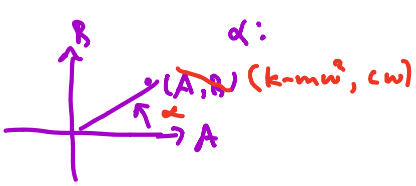
$M\vec{x} = \vec{b}$
$\vec{x} = M^{-1}\vec{b}$.

$$\begin{bmatrix} A \\ B\end{bmatrix} = \frac{1}{(k-m\omega^2)^2 + (c\omega)^2}\begin{bmatrix} k-m\omega^2 & -c\omega \\ c\omega & k-m\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0\end{bmatrix} = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\begin{bmatrix} k-m\omega^2 \\ c\omega \end{bmatrix}$$

$$A\cos\omega t + B\sin\omega t = C\cos(\omega t - \alpha)$$

$$C = \sqrt{A^2+B^2} = \frac{F_0}{(k-m\omega^2)^2 + c^2\omega^2}\sqrt{(k-m\omega^2)^2 + c^2\omega^2} = \frac{F_0}{\sqrt{\underbrace{(k-m\omega^2)^2}_{}+c^2\omega^2}}$$

$$\left(m\left(\tfrac{k}{m}-\omega^2\right)\right)^2$$
$$(m^2)(\omega_0^2-\omega^2)^2$$



α:

$$\cos\alpha = \frac{k-m\omega^2}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}$$

$$\sin\alpha = \frac{c\omega}{\sqrt{(k-m\omega^2)^2 + c^2\omega^2}}$$

Damped forced oscillations ($c > 0$) for $x(t)$:

$$m x'' + c x' + k x = F_0 \cos(\omega t)$$

Undetermined coefficients for $x_P(t)$:

$$\begin{aligned} k\,&\left[x_P = A\cos(\omega t) + B\sin(\omega t)\right] \\ +\, c\,&\left[x_P' = -A\,\omega\sin(\omega t) + B\,\omega\cos(\omega t)\right] \\ +\, m\,&\left[x_P'' = -A\,\omega^2\cos(\omega t) - B\,\omega^2\sin(\omega t)\right] . \end{aligned}$$

$$\begin{aligned} L(x_P) &= \cos(\omega t)\left(k A + c B \omega - m A \omega^2\right) \\ &\quad + \sin(\omega t)\left(k B - c A \omega - m B \omega^2\right) . \end{aligned}$$

Collecting and equating coefficients yields the matrix system

$$\begin{bmatrix} k - m\,\omega^2 & c\,\omega \\ -c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix},$$

which has solution

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 & -c\,\omega \\ c\,\omega & k - m\,\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{\left(k - m\,\omega^2\right)^2 + c^2\omega^2} \begin{bmatrix} k - m\,\omega^2 \\ c\,\omega \end{bmatrix}$$

In amplitude-phase form this reads

$$x_P = A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$

with

$$C = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} . \quad \text{(Check!)}$$

$$\cos(\alpha) = \frac{k - m\,\omega^2}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \longleftarrow m\left(\tfrac{k}{m} - \omega^2\right) = m\left(\omega_0^2 - \omega^2\right)$$

$$\sin(\alpha) = \frac{c\,\omega}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} . \quad > 0$$

And the general solution $x(t) = x_P(t) + x_H(t)$ is given by

- underdamped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p t} C_1 \cos(\omega_1 t - \alpha_1)$.
- critically-damped: $x = x_{sp} + x_{tr} = C\cos(\omega t - \alpha) + e^{-p t}(c_1 t + c_2)$.
- over-damped: $x = x_{sp} + x_{tr} = \underbrace{C\cos(\omega t - \alpha)}_{x_{sp}(t)} + \underbrace{c_1 e^{-r_1 t} + c_2 e^{-r_2 t}}_{x_H = x_{tr}(t)}$.

Important to note:

- The amplitude C in x_{sp} can be quite large relative to $\dfrac{F_0}{m}$ if $\omega \approx \omega_0$ and $c \approx 0$, because the denominator will then be close to zero. This phenomenon is practical resonance.
- The phase angle α is always in the first or second quadrant.

Exercise 4) (a cool M.I.T. video.) Here is practical resonance in a mechanical mass-spring demo. Notice that our math on the previous page exactly predicts when the steady periodic solution is in-phase and when it is out of phase with the driving force, for small damping coefficient c! Namely, for c small, when $\omega^2 << \omega_0^2$ we have α near zero (in phase) for x_{sp}, because $\sin(\alpha) \approx 0, \cos(\alpha) \approx 1$; when $\omega^2 >> \omega_0^2$ we have α near π (out of phase), because $\sin(\alpha) \approx 0, \cos(\alpha) \approx -1$; for $\omega \approx \omega_0$, α is near $\frac{\pi}{2}$, because $\sin(\alpha) \approx 1, \cos(\alpha) \approx 0$.

http://www.youtube.com/watch?v=aZNnwQ8HJHU

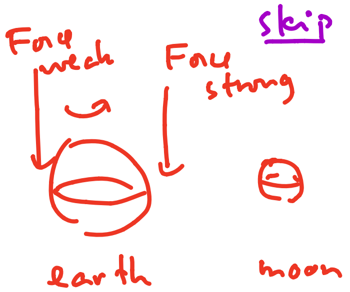
Exercise 5) Solve the IVP for $x(t)$:

$$x'' + 2x' + 26x = 82\cos(4t)$$
$$x(0) = 6$$
$$x'(0) = 0 .$$

Solution:

$$x(t) = \sqrt{41}\cos(4t - \alpha) + \sqrt{10}\, e^{-t}\cos(5t - \beta)$$
$$\alpha = \arctan(0.8),\ \beta = \arctan(-3) .$$

skip



2 high tides/day

forcing period 12 hours

ω_0 ? $T_0 \approx 2$ days

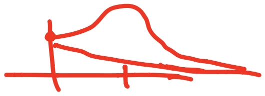
$\omega >> \omega_0$

expect tides to be 180° (π-rad) out of phase.

oops

```
> with(DEtools) :
> dsolve({x''(t) + 2·x'(t) + 26·x(t) = 82·cos(4·t), x(0) = 6, x'(0) = 0});
```

$$x(t) = -3\, e^{-t}\sin(5t) + e^{-t}\cos(5t) + 4\sin(4t) + 5\cos(4t) \qquad (14)$$



Practical resonance: The steady periodic amplitude C for damped forced oscillations is

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}}\,.$$

$C(0) = \frac{F_0}{k}.$

Notice that as $\omega \to 0$, $C(\omega) \to \frac{F_0}{k}$ and that as $\omega \to \infty$, $C(\omega) \to 0$. The precise definition of practical resonance occuring is that $C(\omega)$ have a global maximum greater than $\frac{F_0}{k}$, on the interval $0 < \omega < \infty$.

(Because the expression inside the square-root, in the denominator of $C(\omega)$ is quadratic in ω^2 it will have at most one minimum in the variable ω^2, so $C(\omega)$ will have at most one maximum for non-negative ω. It will either be at $\omega = 0$ or for $\omega > 0$, and the latter case is practical resonance.)

Exercise 6a) Compute $C(\omega)$ for the damped forced oscillator equation related to the previous exercise, except with varying damping coefficient c:

$$x'' + c\,x' + 26\,x = 82\cos(\omega\,t).$$

6b) Investigate practical resonance graphically, for $c = 2$ and for some other values as well. Then use Calculus to test verify practical resonance when $c = 2$.

$c = 0$

$\omega_0 = \sqrt{26}$

$$C(\omega) = \frac{82}{\sqrt{(26-\omega^2)^2 + c^2\omega^2}}$$

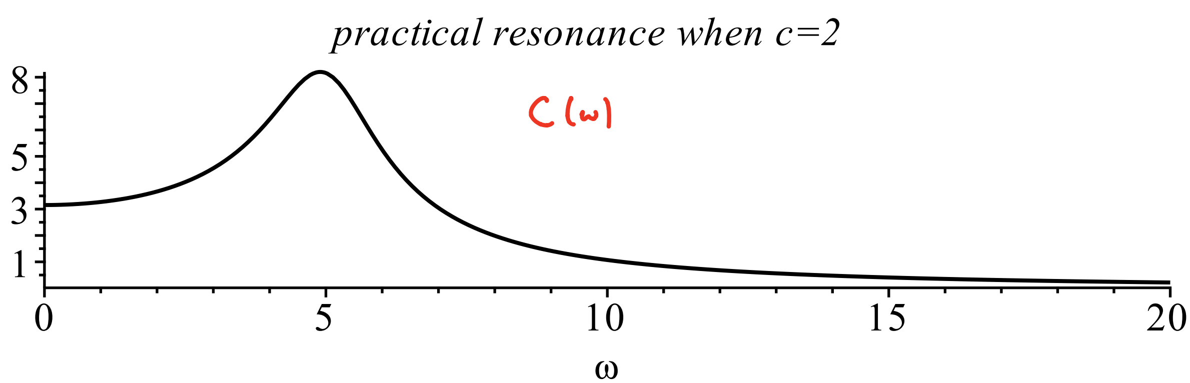
$$= \frac{82}{\sqrt{(26-\omega^2)^2 + 4\omega^2}}$$

> $restart:$

> $with(plots):$

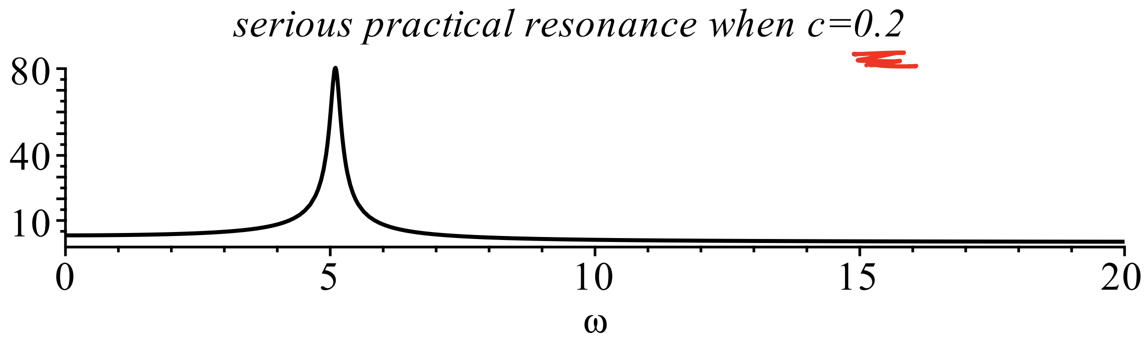
> $C := (\omega, c) \to \dfrac{82}{\sqrt{\left(26 - \omega^2\right)^2 + c^2\cdot\omega^2}}:$

> $plot(C(\omega, 2), \omega = 0..20, color = black, title = \text{`practical resonance when c=2`});$

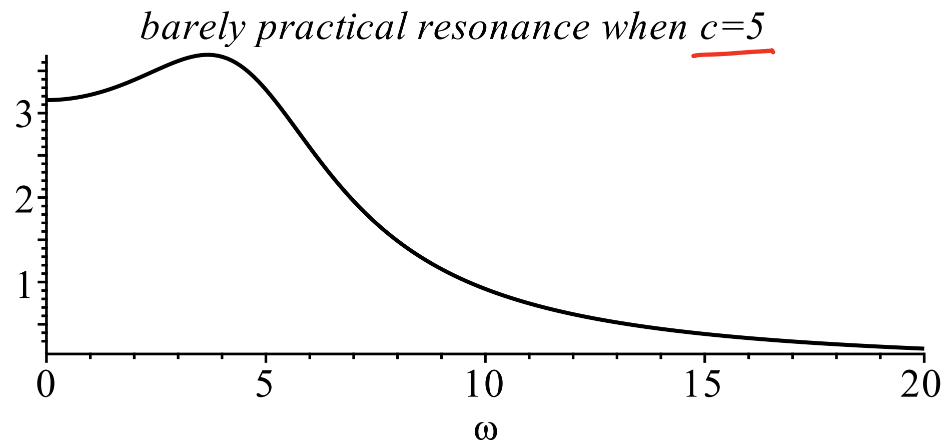


>

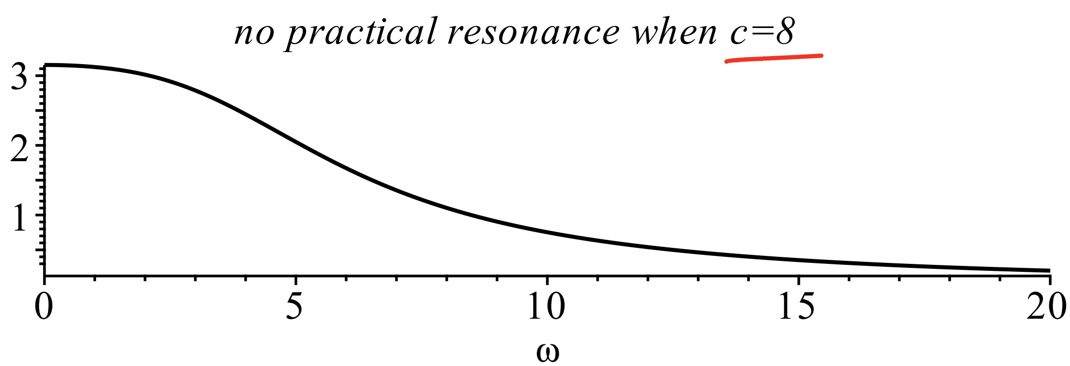
> plot(C(ω, .2), ω = 0 ..20, color = black, title = `serious practical resonance when c=0.2`);



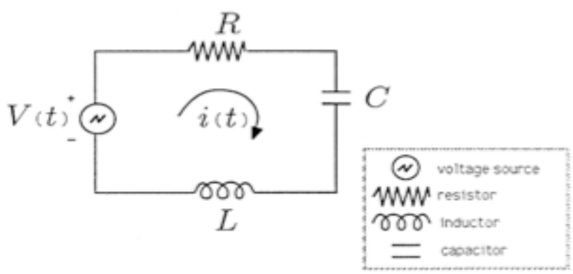
> plot(C(ω, 5), ω = 0 ..20, color = black, title = `barely practical resonance when c=5`);



> plot(C(ω, 8), ω = 0 ..20, color = black, title = `no practical resonance when c=8`);



The mechanical-electrical analogy, continued: Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....recall from earlier in the course:



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $\quad L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t) = E_0 \sin(\omega t)$ 120, 60 typical wall

$\frac{d}{dt}$

For $I(t)$: $\quad L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t) = E_0\,\omega\cos(\omega t)$. $\quad m x'' + c x' + k x = F_0 \cos\omega t$

Transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + \cancel{I_{tr}(t)}^{\to 0} .$$

$$I_{sp}(t) = I_0\cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma)\ , \quad \gamma = \alpha - \frac{\pi}{2} .$$

$x_{sp}(t) = C(\omega)(\cos(\omega t - \alpha))$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \quad \Rightarrow I_0(\omega) = \frac{E_0\cancel{\omega}}{\sqrt{\left(\frac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$=$ $\cancel{\omega}\sqrt{\cancel{\omega^2}\left(\frac{1}{C\omega} - L\omega\right)^2 + R^2\cancel{\omega^2}}$

$$\boxed{\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}}} .$$

The denominator $\sqrt{\left(\frac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the impedance $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\frac{1}{C\,\omega} = L\,\omega$, i.e.

I_0 max when $\frac{1}{C\omega} = L\omega$, $\frac{1}{LC} = \omega^2$

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old radios).}$$

$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable}$$

Both L and C are adjusted in this M.I.T. lab demonstration:
http://www.youtube.com/watch?v=ZYgFuUl9_Vs.

§5.6 forced oscillations.
Fri: no damping — beating $\omega \approx \omega_0$, $\omega \neq \omega_0$
— pure resonance $\omega = \omega_0$
Today Mon: damping — practical resonance
RLC analog

Math 2250-004 Week 11 March 27-31

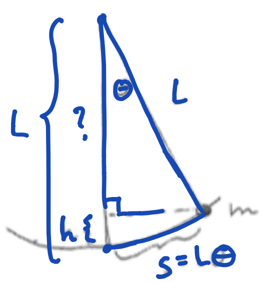
Mon Mar 27: Continue and maybe finish Friday's notes on section 5.6, forced oscillation phenomena

Tues Mar 28: Pendulum and mass-spring experiments from last Tuesday (notes included here, though). Then begin Laplace Transform, 10.1-10.2

Experiment discussion: Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \frac{2\pi}{\omega_0}$ correspond to experimental reality.

Today Tuesday
(1) experiments
(2) RLC circuits from old notes
→ I'll post 2 practice exams & review notes later today (on CANVAS)
Wed: Laplace Transform (not for this exam!)

1) pendulum



$\downarrow \vec{g}$

$\theta = \theta(t)$

$\frac{?}{L} = \cos\theta$

$? = L\cos\theta$

conservative system KE + PE = const.

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$

$h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2}mL^2(\theta'(t))^2 + mgL(1-\cos\theta(t)) \equiv C$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$mL\theta'\underbrace{(L\theta'' + g\sin\theta)}\equiv 0$

$\neq 0$ except at isolated times

⇝ deduce eqtn of motion is

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$$

↘ non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

$\theta < .1$ error if replace $\sin\theta$ by θ is at most $\frac{(.1)^3}{6} = \frac{.001}{6}$

linearize

$$\boxed{x'' + \frac{k}{m}x = 0}$$

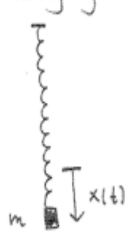
$\omega_0 = \sqrt{\frac{k}{m}}$

$$\boxed{\theta'' + \frac{g}{L}\theta = 0}$$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

2) hanging mass-spring:



$mx'' = -kx$

$mx'' + kx = 0$

$$\boxed{x'' + \frac{k}{m}x = 0}$$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

> *restart* :
> *Digits* := 4 :

> $L := 1.53;$ m
> $g := 9.806;$
> $\omega := \sqrt{\frac{g}{L}}$; # *radians per second*
> $f :=$ *evalf*$(\omega/(2\cdot \mathrm{Pi}))$; # *cycles per second*
> $T := 1/f$; # *seconds per cycle*

$$L := 1.53$$
$$g := 9.806$$
$$\omega := 2.531629974 \quad \omega \quad \text{rad/sec.}$$
$$f := 0.4029214244 \quad \text{cycles/sec.}$$
$$T := \underline{2.481}873486 \quad \text{sec/cycle} \tag{1}$$

Experiment:

10 cycles : 24.3, 24.5, 24.55, 24.8, 24.65, 24.68, 24.4, 24.75, 24.67

$\frac{24.6}{10} \frac{\text{sec}}{\text{cycle}} = \underline{\underline{2.46}}$

Mass-spring:
compute Hooke's constant:

> $98.7 - 83.4$; #*displacement from extra 50g*

$$15.3 \quad / .156 \text{ m} \tag{2}$$

> $k := \frac{.05\cdot 9.806}{.153}$; # *solve k·x=m·g for k.*

$$k := 3.204575163 \quad / \quad k = 3.143 \tag{3}$$

> $m := .1$; # *mass for experiment is 100g*
> $\omega := \sqrt{\frac{k}{m}}$; # *predicted angular frequency*
> $f :=$ *evalf*$\left(\frac{\omega}{2\cdot \mathrm{Pi}}\right)$; # *predicted frequency*
> $T := \frac{1}{f}$; # *predicted period*

$\omega = \sqrt{\frac{3.143}{.1}} = .$

$$m := 0.1$$
$$\omega := 5.660896716 \quad 5.606$$
$$f := 0.9009596945 \quad .892$$
$$T := 1.109927565 \quad \underline{1.12} \tag{4}$$

Experiment:

20 cycles 23 sec, 22.97, 23.38, 23.17, 23.07, ~~23.27~~, 23.27, 23.5

$\frac{23.19}{20} \frac{\text{sec}}{\text{cycle}} = \boxed{1.16 \text{ sec}}$

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>

Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}\,.$$

$$PE_{work} = \int_0^x k\, s\; \mathrm{d}s = \frac{1}{2} k\, x^2, \quad KE_{mass} = \frac{1}{2} m\, (x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\, x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ , as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2} (\mu\, x'(t))^2 \left(m_{spring}\, \mathrm{d}\mu\right)$$

$$= \frac{1}{2} m_{spring} (x'(t))^2 \int_0^1 \mu^2 d\,\mu = \frac{1}{6} m_{spring} (x'(t))^2\ .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3} m_{spring}\right)(x'(t))^2 + \frac{1}{2} k\, x^2 = \frac{1}{2} M (x'(t))^2 + \frac{1}{2} k\, x^2\,,$$

where

$$\boxed{M = m + \frac{1}{3} m_{spring}}$$

$\mathrm{D}_t(TE) = 0 \Rightarrow$

$$M\, x'(t)\, x''(t) + k\, x(t)\, x'(t) = 0\,.$$
$$x'(t)\,(M\, x'' + k\, x) = 0\,.$$

Since $x'(t) = 0$ only at isolated t-values, we deduce that the corrected equation of motion is

$$(M\, x'' + k\, x) = 0$$

with

$$\boxed{\omega_0 = \sqrt{\frac{k}{M}}\ .}$$

Does this lead to a better comparison between model and experiment?

```
> ms := .011;  # spring has mass 11g
  M := m + 1/3·ms;  # "effective mass"
```

$$ms := 0.011$$
$$M := 0.1036666667 \qquad (5)$$

$\sqrt{\frac{3.143}{.1037}}$

$k = 3.143$

> $\omega := \sqrt{\frac{k}{M}}$; # *predicted angular frequency*

$f := evalf\left(\frac{\omega}{2 \cdot \text{Pi}}\right)$; # *predicted frequency*

$T := \frac{1}{f}$; # *predicted period*

$$\omega := 5.559883146$$
$$f := 0.8848828855$$
$$T := 1.130093051 \tag{6}$$

5.506

.876

1.14 sec/cycle

better

>

10.1-10.2 Laplace transform, and application to DE IVPs, including Chapter 5.

linear transformation!

• This new material having to do with the Laplace Transform will help you review and solidify the ideas in Chapter 5. It will not be tested on the second midterm, but will show up on the final exam.

input.

• The Laplace transform is a linear transformation "$\mathcal{L}$" that converts piecewise continuous functions $f(t)$, defined for $t \geq 0$ and with at most exponential growth ($|f(t)| \leq Ce^{Mt}$ for some values of C and M), into functions $F(s)$ defined by the transformation formula

output

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^{\infty} f(t)e^{-st}\, dt .$$

$|f(t)|e^{-st} \leq Ce^{Mt}e^{-st} = Ce^{(M-s)t}$ decays $s>M$

• Notice that the integral formula for $F(s)$ is only defined for sufficiently large s, and certainly for $s > M$, because as soon as $s > M$ the integrand is decaying exponentially, so the improper integral from $t = 0$ to ∞ converges.

• The convention is to use lower case letters for the input functions and (the same) capital letters for their Laplace transforms, as we did for $f(t)$ and $F(s)$ above. Thus if we called the input function $x(t)$ then we would denote the Laplace transform by $X(s)$.

Taking Laplace transforms seems like a strange thing to do. And yet, the Laplace transform $\mathcal{L}$ is just one example of a collection of useful "integral transforms". $\mathcal{L}$ is especially good for solving IVPs for linear DEs, as we shall see starting today. Other famous transforms - e.g. Fourier series and Fourier transform are extremely important in studying linear partial differential equations, as you will see in e.g. Math 3140, 3150, physics, engineering, or Wikipedia.

Exercise 1) Use the definition of Laplace transform

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^{\infty} f(t)e^{-st}\, dt$$

to check the following facts, which you will also find inside the front cover of your text book.

a) $\mathcal{L}\{1\}(s) = \dfrac{1}{s} \quad (s > 0)$

b) $\mathcal{L}\{e^{\alpha t}\}(s) = \dfrac{1}{s-\alpha} \quad (s > \alpha \text{ if } \alpha \in \mathbb{R},\ s > a \text{ if } \alpha = a + k\,i \in \mathbb{C})$

c) Laplace transform is linear, i.e.

$$\mathcal{L}\{f_1(t) + f_2(t)\}(s) = F_1(s) + F_2(s) .$$
$$\mathcal{L}\{c f(t)\}(s) = c F(s).$$

d) Use linearity and your work above to compute $\mathcal{L}\{3 - 4e^{-2t}\}(s)$.

a). $\mathcal{L}\{1\}(s) = \int_0^{\infty} 1\, e^{-st}\, dt = \left.\dfrac{e^{-st}}{-s}\right]_{t=0}^{\infty} = \dfrac{e^{(-s)\cdot\infty}}{-s} - \dfrac{1}{-s} = 0 + \dfrac{1}{s} = \boxed{\dfrac{1}{s}}$

$s>0$, term $=0$

b) $\mathcal{L}\{e^{\alpha t}\}(s) = \int_0^{\infty} e^{\alpha t}e^{-st}\, dt = \int_0^{\infty} e^{(\alpha-s)t}\, dt = \left.\dfrac{e^{(\alpha-s)t}}{\alpha-s}\right]_{t=0}^{\infty} = 0 - \dfrac{1}{\alpha-s} = \boxed{\dfrac{1}{s-\alpha}}$

$s>\alpha$
$s>\text{Re}(\alpha)$

c) $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\}(s)$

$$= \int_0^\infty \underbrace{\left(c_1 f_1(t) + c_2 f_2(t)\right) e^{-st}}_{c_1 f_1(t) e^{-st} + c_2 f_2(t) e^{-st}} dt = c_1 \int_0^\infty \cancel{c_1} f_1(t) e^{-st} dt + c_2 \int_0^\infty \cancel{c_2} f_2(t) e^{-st} dt$$

$$= c_1 F_1(s) + c_2 F_2(s)$$

$$\downarrow$$

$$\mathcal{L}\{f_1(t)\}(s)$$

d). $\mathcal{L}\{3 - 4e^{-2t}\}(s) = 3\mathcal{L}\{1\}(s) - 4\mathcal{L}\{e^{-2t}\}(s)$

$$= \frac{3}{s} - 4\frac{1}{s--2} = \frac{3}{s} - \frac{4}{s+2}$$

For the DE's like we've just been studying in Chapter 5, the following Laplace transforms are very important:

Exercise 2) Use complex number algebra, including Euler's formula, linearity, and the result from 1b that

$$\mathcal{L}\{e^{(a+k\,i)t}\}(s) = \frac{1}{s - (a + k\,i)}$$

to verify that

a) $\mathcal{L}\{\cos(k\,t)\}(s) = \dfrac{s}{s^2 + k^2}$ $\overset{\text{by hand}}{=} \int_0^\infty (\cos kt) e^{-st} dt$

b) $\mathcal{L}\{\sin(k\,t)\}(s) = \dfrac{k}{s^2 + k^2}$

c) $\mathcal{L}\left\{e^{a\,t}\cos\left(k\,t\right)\right\}(s) = \dfrac{s - a}{(s-a)^2 + k^2}$. $\qquad r = a \pm ki$

d) $\mathcal{L}\{e^{a\,t}\sin(k\,t)\}(s) = \dfrac{k}{(s-a)^2 + k^2}$.

(Notice that if we tried doing these Laplace transforms directly from the definition, the integrals would be messy but we could attack them via integration by parts or integral tables.)

$$\mathcal{L}\{e^{ikt}\}(s) = \frac{1}{s - ik} \qquad \alpha = ik.$$

$$= \mathcal{L}\{\cos kt + i \sin kt\}(s)$$

$$= \boxed{\mathcal{L}\{\cos kt\}(s)} + i\,\mathcal{L}\{\sin kt\}(s) = \frac{1}{(s-ik)}\left(\frac{s+ik}{s+ik}\right) = \frac{s+ik}{s^2+k^2}$$

$$= \boxed{\frac{s}{s^2+k^2}} + i\,\frac{k}{s^2+k^2}$$

c, d.) $\mathcal{L}\{e^{(a+ki)t}\}(s) = \dfrac{1}{s-(a+ki)} = \dfrac{1}{(s-a)-ik}$

$$= \frac{1}{(s-a)-ik}\,\frac{(s-a)+ik}{(s-a)+ik} = \frac{s-a+ik}{(s-a)^2+k^2}$$

$$\mathcal{L}\{e^{at}\cos kt + i e^{at}\sin kt\}(s)$$

$$\mathcal{L}\{e^{at}\cos kt\}(s) + i\,\mathcal{L}\{e^{at}\sin kt\}(s) = \frac{s-a}{(s-a)^2+k^2} + i\,\frac{k}{(s-a)^2+k^2}$$

It's a theorem (hard to prove but true) that a given Laplace transform $F(s)$ can arise from at most one piecewise continuous function $f(t)$. (Well, except that the values of f at the points of discontinuity can be arbitrary, as they don't affect the integral used to compute $F(s)$.) Therefore you can read Laplace transform tables in either direction, i.e. not only to deduce Laplace transforms, but inverse Laplace transforms $\mathcal{L}^{-1}\{F(s)\}(t) = f(t)$ as well.

Exercise 3) Use the Laplace transforms we've computed and linearity to compute

$$\mathcal{L}^{-1}\left\{\frac{7}{s} + \frac{1}{4}\frac{4}{s^2+16} - \frac{10\,s}{s^2+16}\right\}(t)\,.$$

$$= 7 + \frac{1}{4}\sin 4t - 10\cos 4t$$

$\mathcal{L}\{f(t)\}(s) = 0$

only soltn is

$f_H(t) = 0$

$\mathcal{L}^{-1}$

$\mathcal{L}$

$f(t)$ $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
1	$\frac{1}{s}$ $(s > 0)$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(\alpha))$
$\cos(k\,t)$	$\frac{s}{s^2+k^2}$ $(s > 0)$
$\sin(k\,t)$	$\frac{k}{s^2+k^2}$ $(s > 0)$
$e^{a\,t}\cos(k\,t)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$
$e^{a\,t}\sin(k\,t)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$
$f'(t)$	$s\,F(s) - f(0)$
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$

$\leftarrow k = 4$

$\sin 4t \;\Big|\; \frac{4}{s^2+16}$

$\mathcal{L}\{g'(t)\}(s) = s\mathcal{L}\{g(t)\}(s) - g(0)$

$\mathcal{L}\{f''(t)\}(s) = s\mathcal{L}\{f'(t)\}(s) - f'(0)$

let $g(t) = f'(t)$

$\mathcal{L}\{f''(t)\}(s) = s\left(s\,F(s) - f(0)\right) - f'(0)$

$= s^2F(s) - s\,f(0) - f'(0)$

Laplace transform table

The integral transforms of DE's and PDE's were designed to have the property that they convert the corresponding linear DE and PDE problems into algebra problems. For the Laplace transform it's because of these facts:

Exercise 4a) Use integration by parts and the definition of Laplace transform to show that

$$\mathcal{L}\{f'(t)\}(s) = s\,\mathcal{L}\{f(t)\}(s) - f(0) = s\,F(s) - f(0).$$

4b) Use the result of a, applied to the function $f'(t)$ to show that

$$\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s\,f(0) - f'(0).$$

4c) What would you guess is the Laplace transform of $f'''(t)$? Could you check this?

4a) $\mathcal{L}\{f'(t)\}(s) = \int_0^\infty f'(t)e^{-st}\,dt = uv\Big]_0^\infty - \int_0^\infty v\,du$

$u = e^{-st}$; $du = -se^{-st}\,dt$

$dv = f'(t)dt$; $v = f(t)$

$|f(t)| \le Ce^{Mt}$

$s > M$

$= e^{-st}f(t)\Big]_{t=0}^\infty - \int_0^\infty f(t)(-s)e^{-st}\,dt$

$s > M$

$= 0 - f(0) + s\int_0^\infty f(t)e^{-st}\,dt$

$= sF(s) - f(0)$

Here's an example of using Laplace transforms to solve DE IVPs, in the context of Chapter 5 and the mechanical (and electrical) application problems we just considered there.

Exercise 5) Consider the undamped forced oscillation IVP

$$x''(t) + 4\,x(t) = 10\cos(3\,t)$$
$$x(0) = 2$$
$$x'(0) = 1$$

If $x(t)$ is the solution, then both sides of the DE are equal. Thus the Laplace transforms are equal as well... .so, equate the Laplace transforms of each side and use algebra to find $\mathcal{L}\{x(t)\}(s) = X(s)$. Notice you've computed $X(s)$ without actually knowing $x(t)$! If you were happy to stay in "Laplace land" you'd be done. In any case, at this point you can use our table entries to find $x(t) = \mathcal{L}^{-1}\{x(t)\}(s)$.

(Notice that if your algebra skills are good you've avoided having to use the Chapter 5 algorithm of (i) find x_H (ii) find an x_P (iii) $x = x_P + x_H$ (iv) solve IVP.) Magic!

on exam

$x_H = c_1\cos 2t + c_2\sin 2t$

$x_p = A\cos 3t$

$x = A\cos 3t + c_1\cos 2t + c_2\sin 2t$

solve IVP!

$\mathcal{L}$: DE is true for soln $x(t)$

so $\mathcal{L}\{x''(t) + 4x(t)\}(s) = \mathcal{L}\{10\cos 3t\}(s)$

$s^2X(s) - s\underbrace{x(0)}_{2} - \underbrace{x'(0)}_{1} + 4X(s) = 10\dfrac{s}{s^2+9}$

$X(s)(s^2+4) = \dfrac{10s}{s^2+9} + 2s + 1$

$X(s) = \dfrac{10s}{(s^2+4)(s^2+9)} + \dfrac{2s}{s^2+4} + \dfrac{1}{s^2+4}$

$$X(s) = 10s\left(\frac{1}{(s^2+4)(s^2+9)}\right) + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$= \frac{\overset{2}{\cancel{10}}s}{\cancel{5}}\left(\frac{1}{s^2+4} - \frac{1}{s^2+9}\right) + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$X(s) = \frac{2s}{s^2+4} - \frac{2s}{s^2+9} + \frac{2s}{s^2+4} + \frac{1}{s^2+4}$$

$$X(s) = \frac{4s}{s^2+4} + \frac{1}{s^2+4} - \frac{2s}{s^2+9}$$

$$\boxed{x(t) = 4\cos 2t + \tfrac{1}{2}\sin 2t - 2\cos 3t}$$

> *with*(*DEtools*) :

> *dsolve*($\{x''(t) + 4\cdot x(t) = 10\cdot\cos(3\cdot t), x(0) = 2, x'(0) = 1\}$); # *to check*

$$\boxed{x(t) = \frac{1}{2}\sin(2t) + 4\cos(2t) - 2\cos(3t)} \qquad \textbf{(7)}$$

Exercise 6) Use Laplace transform as above, to solve the IVP for the following underdamped, unforced oscillator DE:

$$x''(t) + 6x'(t) + 34x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

> *dsolve*($\{x''(t) + 6\cdot x'(t) + 34\cdot x(t) = 0, x(0) = 3, x'(0) = 1\}$); # *to check*

$$x(t) = 2e^{-3t}\sin(5t) + 3e^{-3t}\cos(5t) \qquad \textbf{(8)}$$

>

Wed : • Laplace transforms in Tuesday notes (HW due next Thurs. on Laplace)
(filled in review sheet is posted in CANVAS, and in today's notes)
• last example today ties into midterm

Wed Mar 29
Finish Tuesday notes on Laplace transform introduction, and then review for Friday midterm exam.

Exam 2 Review Questions
Math 2250-004 March 2017

Our exam covers chapters 3.6, 4.1-4.4, 5.1-5.6 of the text. Only scientific calculators will be allowed on the exam.

• remember to get to class a few minutes early. The exam will be from 10:40 - 11:40 a.m. Graphing calculators are not allowed, only scientific ones. Symbolic answers are allowed, although a scientific calculator could give you confidence on an amplitude-phase calculation, for example.

I try to find problems that touch on most of these key topics. I've put $*$'s next to topics which have higher probabilities of appearing on my exams, although anything we've learned is fair game.

Chapter 3.6: Determinants. Approximately 10% of the exam could be related to this material.

Be able to compute |A| for a square matrix A using cofactor expansions, row operations, or some combination of those procedures.

* What does the value of |A| have to do with whether A^{-1} exists?

$|A| \neq 0 \iff \operatorname{rref}(A) = I \iff A^{-1}$ exists $\iff$ solns $\vec{x}$ to $A\vec{x} = \vec{b}$ always exist & are unique

What's the magic formula for the inverse of a matrix? Can you work with this formula in the two by two or three by three cases? Can you use it?

$$A^{-1} = \frac{1}{|A|}\operatorname{Adj}(A) = \frac{1}{|A|}(\operatorname{cof}(A))^T$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Chapter 4.1-4.4

Approximately 40% of the exam will deal directly with this material....but much of Chapter 5 uses these concepts, so much more than 40% of the exam will be related to chapter 4. (And, as far as matrix and determinant computations go, you should remember everything you learned in Chapter 3.)

* **Do you know the key definitions?**

vector space : a collection of objects "V" together with "$+$" vector addition & "$\cdot$" scalar multiplication such that

α) $f, g \in V \Rightarrow f+g \in V$

β) $f \in V,\ c \in \mathbb{R} \Rightarrow cf \in V$

AND so that a long list of algebra properties (that we know to be true for vector addition & scalar mult. in $\mathbb{R}^n$ and for functions are true)

a **linear combination** of a collection $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k\}$ of vectors is any sum of scalar multiples of them, i.e. any

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k$$

linearly independent vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k$

$\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\}$ is linearly independent if and only if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0} \iff c_1 = c_2 = \ldots = c_n = 0$$

linearly dependent vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k$

not independent. i.e. some $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k = \vec{0}$ where not all c_j's $= 0$

span of a collection of vectors $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_k\}$

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k\} = \text{the collection of all linear combinations} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_k\vec{v}_k \mid c_1, c_2, \ldots c_k \in \mathbb{R}\}$$

subspace of a vector space V

is a subset "W" that is closed under addition & scalar multiplication

i.e. α) $f, g \in W \Rightarrow f+g \in W$

β) $f \in W,\ c \in \mathbb{R} \Rightarrow cf \in W$

(so W is a sub vector space of V)

basis of a vector space "V"

a collection of vectors $\{f_1, f_2, \ldots f_k\}$ in V so that

- it spans V (i.e. $span\{f_1, f_2, \ldots f_k\} = V$)
- $\{f_1, f_2, \ldots f_n\}$ is linearly independent

dimension of a vector space V

the number of vectors in a basis for V

(more vectors will be linearly dependent
fewer vectors will fail to span V)

- **Subspace examples from Chapter 4**, involving the concepts above

  - the space of solutions $\underline{\boldsymbol{x}}$ to matrix equations $[A]\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$

    If $A_{m\times n}$ then solutions $\vec{x}$ to $A\vec{x} = \vec{0}$ are in $\mathbb{R}^n$

  - $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_k\}$

    if $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \in \mathbb{R}^m$
    their span is a subspace of $\mathbb{R}^m$

- **Subspace examples from Chapter 5**

  - solution space to homogeneous linear differential equation for e.g. $y = y(x)$ on an interval I, i.e. solutions to

$$L(y) := y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0$$

implicit description

  - $span\{y_1, y_2, \ldots y_n\}$

* **What does it mean for a transformation $L: V \rightarrow W$ between vector spaces to be linear?**

L takes sums to sums & scalar multiples to scalar multiples

i.e.

(i) $L(y_1 + y_2) = L(y_1) + L(y_2)$

(ii) $L(cy) = c\,L(y)$

hold for all $y_1, y_2, y \in V$, $c \in \mathbb{R}$

Chapter 4 examples?

$L(\vec{x}) := A\vec{x}$. if $A_{m \times n}$ then L transforms $\mathbb{R}^n$ into a subspace of $\mathbb{R}^m$ (spanned by the columns of A)

the set of vectors that are transformed into $\vec{0}$, i.e. the solns $\vec{x}$ to $A\vec{x} = \vec{0}$, are a subspace of $\mathbb{R}^n$

Chapter 5 examples?

for $y(x)$ $L(y) = y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_1 y' + p_0 y$

[analogous for $x(t)$]

* **What is the general solution to $L(y) = f$, if L is a linear transformation (or "operator"), in terms of particular and homogeneous solutions? Can you explain why?**

$y = y_P + y_H$ where y_P is any single particular sol'n to $L(y_P) = f$ (the inhomogeneous problem) & y_H is the general soln to the homog. problem, $L(y) = 0$.

Why? if $L(y_P) = f$ & $L(y_H) = 0$

then $L(y_P + y_H) = L(y_P) + L(y_H)$
$= f + 0$
$= f$

if also $L(y_q) = f$

then $y_q = y_P + (y_q - y_P)$

and $L(y_q - y_P) = L(y_q) - L(y_P)$
$= f - f = 0$

so $y_q - y_P$ is some homogeneous sol'n

Chapter 5

About 50% of the exam will be related to this material.

* What is the **natural initial value problem** for n^{th} - order linear differential equation, i.e. the one that has unique solutions?

$$\text{IVP}\left\{\begin{array}{c} L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = f(x) \\ y(x_0) = b_0 \\ y'(x_0) = b_1 \\ \vdots \\ y^{(n-1)}(x_0) = b_{n-1} \end{array}\right.$$

* What is the **dimension of the solution space to the homogeneous DE**

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 \ ?$$

Why? dimension $= n$ because the n initial conditions uniquely determine a solution $y(x)$

* How can you tell if $\{y_1(x), y_2(x), \ldots y_n(x)\}$ is a **basis** for space of solutions to the homogeneous DE above?

If each IVP for $L(y) = 0$ has a unique linear combo soltn: $y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$

$$\left\{\begin{array}{l} c_1 y_1(x_0) + c_2 y_2(x_0) + \ldots + c_n y_n(x_0) = b_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) + \ldots + c_n y_n'(x_0) = b_1 \\ \vdots \\ c_1 y_1^{(n-1)}(x_0) + \ldots + c_n y_n^{(n-1)}(x_0) = b_{n-1} \end{array}\right.$$

How is your answer above related to a **Wronskian matrix** and the **Wronskian determinant**?

iff $\det(W(y_1, y_2, \ldots y_n)) \neq 0$ (at x_0)

since that system is the matrix system

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & y_n'(x_0) \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)}(x_0) & & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

and if Wronskian det $\neq 0$ at x_0 there exists unique $\vec{c}$ solving the matrix system.

* How do you find the **general solution** to the **homogeneous constant coefficient linear DE**

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0\ ?$$

(your answer should involve the characteristic polynomial, Euler's formula, repeated roots, complex roots.) (Do you remember Euler's formula? Can you use it in the various ways we've seen? Do you remember the Taylor-Maclaurin series formula in general? For e^x, $\cos(x)$, $\sin(x)$ in particular?)

try $y = e^{rx} \Rightarrow L(y) = e^{rx}\underbrace{(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0)}_{p(r) \text{ characteristic poly.}} = 0$

get n linearly independent soltns $y_1, y_2, \ldots y_n$ & then $y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$

- roots r_j of p yield soltns $e^{r_j x}$
- double roots also yield $x e^{r_j x}$ etc.
- complex roots $r_j = a \pm ib$ yield complex exponential soltns. Via Euler $e^{i\theta} = \cos\theta + i\sin\theta$ extract real soltns $e^{ax}\cos bx$, $e^{ax}\sin bx$ etc.

Do you know how to find **particular solutions to constant coefficient non-homogeneous linear DEs**

$$L(y) := a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f?$$

* using undetermined coefficients

look for smallest subspace W, with $f \in W$ & so that $L: W \to W$

case 1 no homogeneous soltns in W except for zero fcn then get unique $y_P \in W$

case 2 : part of your guess for y_P is homogeneous soltn. multiply guess by $x, x^2, \ldots x^s$ smallest s so that no term is a homogeneous soln

* Can you solve IVP's for solutions $y(x)$ to

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x)$$

$$y(x_0) = b_0$$
$$y'(x_0) = b_1$$
$$\vdots$$
$$y^{(n-1)}(x_0) = b_{n-1}$$

- find y_H
- find a y_P
- set $y = y_P + y_H = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$
- solve IVP by solving the resulting system for $c_1, c_2, \ldots c_n$.

5.4, 5.6, EP3.7 Mechanical vibrations and forced oscillations; electrical circuit analog

∗ What is the governing second order DE's for a **damped mass-spring configuration** (via Newton's second law)? units? for $x(t)$:

$$mx'' + cx' + kx = F(t)$$

each term has force units ($N = \frac{kg\,m}{s^2}$ in m-k-s)

mass : kg

c : $N\frac{s}{m}$ or $\frac{kg}{s}$

k: N/m

∗ What case of the governing DE leads to **unforced damped oscillations?** What four phenomena have we discussed for unforced oscillation problems, and how to they arise? (Hint: one is undamped; three are damped.)

$$mx'' + cx' + kx = 0 \qquad p(r) = mr^2 + cr + k$$

- simple harmonic motion $(c = 0)$ $p(r)$ imaginary roots
- damping, $c > 0$
  - overdamped $p(r)$ roots $r_1 < r_2 < 0$
  - critically damped. double root $r = r_1 < 0$
  - underdamped $p(r) = -a \pm \omega_1 i$

∗ What are the forms of the solutions in these four cases?

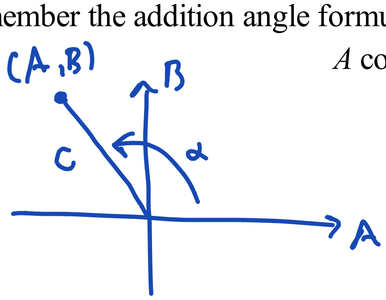
$c = 0$: $x_H = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t = C\cos(\omega_0 t - \alpha)$ $\omega_0 = \sqrt{\frac{k}{m}}$

$c > 0$

- over-damped $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $r_1, r_2 < 0$
- crit-damped $x = ce^{r_1 t} + c_2 t e^{r_1 t}$
- under-damped $x = e^{-at}(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t)$
  $= e^{-at}(C\cos(\omega_1 t - \alpha_1))$

∗ Can you convert a linear combination $A\cos(\omega t) + B\sin(\omega t)$ in amplitude-phase form? Do you remember the addition angle formulas? Can you explain the physical properties of the solution?

$$A\cos(\omega t) + B\sin(\omega t) = C\cos(\omega t - \alpha)$$



$$C = \sqrt{A^2 + B^2}$$

$$\cos\alpha = \frac{A}{C}$$

$$\sin\alpha = \frac{B}{C}$$

$$\tan\alpha = \frac{B}{A}$$

* What are the possible phenomena with **forced undamped oscillations** (assuming the forcing function is sinusoidal) and how do they arise?

$$m\,x''(t) + k\,x(t) = F_0\cos(\omega\,t) \text{ or } F_0\sin(\omega\,t)$$

$x_p = A(\cos\omega t - \cos\omega_0 t)$

- beating $\omega \approx \omega_0 = \sqrt{\frac{k}{m}}$, $\omega \neq \omega_0$.
- pure resonance $\omega = \omega_0$ $\quad x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t)$

* What are the possible phenomena with **forced damped oscillations** (assuming the forcing function is sinusoidal)?

$$m\,x''(t) + c\,x'(t) + k\,x(t) = F_0\cos(\omega\,t)$$

- practical resonance

$x_p(t) = \underbrace{x_{sp}}_{\text{steady periodic}} = C\cos(\omega t - \alpha)$

with $\omega \approx \omega_0$
and c small enough
amplitude $C = C(\omega)$
is "large"
($C(\omega)$ has a max value at some $\omega^* > 0$)

* Can you solve all initial value problems that arise in the situations above?

yes.
I can. ☺ can you?

- find x_H
- find x_p
- use $x = x_p + x_H$ to solve IVP.

For deeper review, consult class notes and examples, quizzes, and homework+lab problems.