Friday
• start details of today's notes

Monday
: Finish Fri. notes
• talk about pendulum model

Tuesday
• the experiments
• start Laplace transforms

Wed
• Laplace
• review notes

Math 2250-004
Friday Mar 24

tomorrow Thurs in labs: $y(x) \longrightarrow x(t)$

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on Monday using these notes.

Overview for solutions $x(t)$ to

$$\boxed{m\,x'' + c\,x' + k\,x = F_0 \cos(\omega\,t)}$$

using section 5.5 undetermined coefficients algorithms.

• **undamped** $(c = 0)$:

In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

$$m x'' + k x = F_0 \cos \omega t$$
$$x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t$$
$$x'' + \omega_0^2 x = \frac{F}{m} \cos \omega t$$

which has simple harmonic motion solutions $x_H(t) = C\cos(\omega_0 t - \alpha)$. So for the non homongeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

$x_p = A \cos \omega t + \cancel{B \sin \omega t}$

• $\omega \neq \omega_0 := \sqrt{\dfrac{k}{m}} \Rightarrow x_P = A\cos(\omega\,t)$ because only even derivatives, we don't need $\sin(\omega\,t)$ terms !!

$$\boxed{\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0\cos(\omega_0 t - \alpha_0)}$$

• $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!

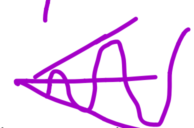
• $\omega = \omega_0$ $\quad \Rightarrow \boxed{x_P = t\left(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\right)}$

$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega_0\,t - \alpha) + C_0\cos(\omega_0 t - \alpha_0).$$

("pure" resonance!)

$x'' + \omega_0^2 x = \frac{F}{m} \cos \omega_0 t$

pure reson.



....

$$\boxed{m\,x'' + c\,x' + k\,x = F_0\cos(\omega\,t)}$$

• **damped** $(c > 0)$: in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm i\,\omega$ when $c > 0$).

• underdamped: $\quad x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)$.
• critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)$.
• over-damped: $\quad x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

steady periodic part of the soln

$x_{sp}(t) = x_p(t)$

$x_H(t) \rightarrow 0$ exp. $\quad x_H(t) = x_{tr}(t)$ transient part.

take-home quiz
hand in at labs tomorrow

- in all three damped cases on the previous page, $x_H(t) \to 0$ exponentially and is called the transient solution $x_{tr}(t)$ (because it disappears as $t \to \infty$. And in these damped cases $x_P(t)$ as above is called the steady periodic solution $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit practical resonance. This can be an important phenomenon in electrical circuits, where amplifying signals is important.

forced undamped oscillations:

Exercise 1a) Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0.$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. What is this (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

$mx'' + kx = F_0 \cos \omega t$

$x'' + \underbrace{\frac{k}{m}}_{\omega_0^2} x = \frac{F_0}{m} \cos \omega t$

x_H : $x'' + 9x = 0$

$x_H(t) = A\cos 3t + B\sin 3t$

$(p(r) = r^2 + 9 = 0$
$r^2 = -9$
$r = \pm 3i)$

$+9\,[\,x_p = C\cos(5t)\,]$
$+0\,[\,x' = -5C\sin(5t)\,]$
$1\,[\,x'' = -25C\cos 5t\,]$

$x'' + 9x = \cos 5t\,(\underbrace{9C - 25C}_{-16C}) \overset{\text{want}}{=} 80\cos 5t$

need $-16C = 80$
$C = -5$

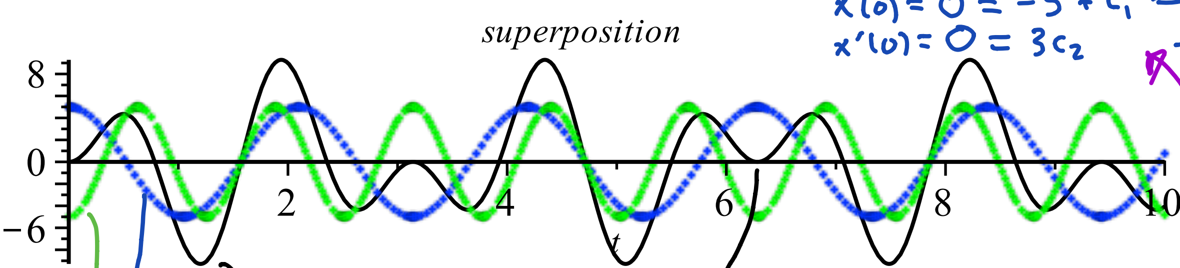
$\boxed{x_p = -5\cos 5t}$

$\boxed{x(t) = -5\cos 5t + 5\cos 3t}$

$x(0) = 0$
$x'(0) = 0$ ✓

$x(t) = x_p + x_H$
$x(t) = -5\cos 5t + c_1\cos 3t + c_2\sin 3t$
$x(0) = 0 = -5 + c_1 \Rightarrow c_1 = 5$
$x'(0) = 0 = 3c_2 \Rightarrow c_2 = 0$

```
> with(plots):
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point):
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point):
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black):
  display({plot1, plot2, plot3}, title='superposition');
```



x_p x_H $x(t)$

$\approx 6.2 = 2\pi$

$(x'(t) = +25\sin 5t - 3c_1\sin 3t + 3c_2\cos 3t$
$x(0) = 0 + 0 + 3c_2)$

period of soln $x(t)$?

$x(t) = -5\cos 5t + 5\cos 3t$

$T_1 = \frac{2\pi}{5}$ $T_2 = \frac{2\pi}{3}$

least common multiple $(\frac{2\pi}{5}, \frac{2\pi}{3}) = 2\pi$

$\cdot 15:\quad n_1\left(\cancel{\frac{2\pi}{5}}\right) = n_2\left(\frac{2\pi}{3}\right) \qquad n_1, n_2 \in \mathbb{N}$

$3n_1 = 5n_2$

$n_1 = 5$
$n_2 = 3$

$5\left(\frac{2\pi}{5}\right)$ or $3\left(\frac{2\pi}{3}\right)$

2π

In general:

undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

$$\begin{aligned}
+\tfrac{k}{m}\,(x_p &= A\cos\omega t) \\
+0\,(x_p' &= -A\omega\sin\omega t) \\
+1\,(x_p'' &= -A\omega^2\cos\omega t) \\
\hline
L(x_p) &= \cos\omega t\, A\left[\underbrace{\tfrac{k}{m}}_{\omega_0^2} - \omega^2\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega t
\end{aligned}$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$. $\qquad A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$

$A = \frac{F_0}{m}(\omega_0^2-\omega^2)$

$$\boxed{\text{so, } x_p(t) = -\frac{F_0}{m}\frac{1}{(\omega^2-\omega_0^2)}\cos\omega t}$$

Note $x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

$x = x_p + x_H$

$x = -\frac{F_0}{m(\omega^2-\omega_0^2)}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t$

$x(0) = x_0 = -\frac{F_0}{m(\omega^2-\omega_0^2)} + c_1$

$\boxed{c_1 = x_0 + \frac{F_0}{m(\omega^2-\omega_0^2)}}$

$x'(0) = v_0 = \omega_0 c_2$

$\boxed{c_2 = \frac{v_0}{\omega_0}}$

so, by plugging in or observation
IVP solution is

$$\boxed{x(t) = \frac{+F_0}{m(\omega^2-\omega_0^2)}\left(\underbrace{\cos\omega_0 t}_{x_H} - \underbrace{\cos\omega t}_{x_p}\right) + \underbrace{x_0\cos\omega_0 t}_{x_H} + \underbrace{\frac{v_0}{\omega_0}\sin\omega_0 t}_{x_H}}$$

check WR!

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned}
\cos(\underbrace{\alpha - \beta}_{\omega_0 t}) - \cos(\underbrace{\alpha + \beta}_{\omega t}) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\
&\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\
&= 2\sin(\alpha)\sin(\beta)\ .
\end{aligned}$$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t,\ \ \beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$T_1 = \frac{2\pi}{\bar{\omega}} \qquad T_2 = \frac{2\pi}{\frac{1}{2}|\omega-\omega_0|} = \frac{4\pi}{|\omega-\omega_0|}$

$$x(t) = \frac{F_0}{m\left(\omega^2 - \omega_0^2\right)} 2\sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \qquad \text{period: } \frac{4\pi}{|\omega - \omega_0|}\ .$$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

as $\omega \to \omega_0$
get resonance formula

$$x(t) \longrightarrow \frac{F_0\ \cancel{2}}{m\cancel{(\omega-\omega_0)}(\omega+\omega_0)}(\sin\omega_0 t)\left[\cancel{\tfrac{1}{2}}\cancel{(\omega-\omega_0)}t \pm \frac{\theta^3}{3!}\right]$$

$$\longrightarrow \boxed{\frac{F_0}{2m\omega_0}t\sin\omega_0 t}$$

$$x(t) = \frac{F_0}{m(\omega^2-\omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) = \frac{F_0}{m(\omega^2-\omega_0^2)} 2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

We will call half that period the beating period, as explained by the next exercise: $(\text{when } x_0 = 0,\ v_0 = 0)$

$$\text{beating period: } \frac{2\pi}{|\omega-\omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m\left|\omega^2-\omega_0^2\right|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\,t) \qquad x'' + \omega_0^2 x = \frac{F_0}{m}\cos(3.1t)$$
$$\left.\begin{aligned} x(0) &= 0 \\ x'(0) &= 0. \end{aligned}\right\}$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

$\frac{F_0}{m} = 80$

$\omega_0 = 3$

$\omega = 3.1$

$(3.1)^2 - 3^2 = (3.1-3)(3.1+3)$

$$x(t) = 80\frac{1}{.61}\left(\cos 3t - \cos 3.1t\right)$$

$$= \left(\frac{80}{.61}\cdot 2\right)\sin(3.05t)\,\underbrace{\sin(.05t)}$$

$$T_2 = \frac{2\pi}{.05} = 40\pi \approx 120$$

beating period $= 20\pi$!

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> plot(262.3·sin(3.05·t)sin(.05·t), t=0..100, color=black, title=`beating`);
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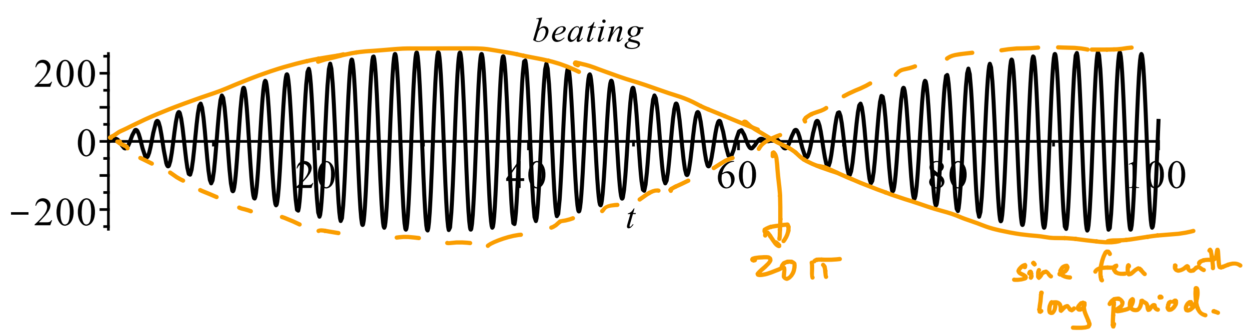


<u>Resonance:</u>

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$\begin{aligned}
+\omega_0^2 (\quad & x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) &)\\
0\,(\quad & x_p' = t\,(-A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t &)\\
+1\,(\quad & x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 &)
\end{aligned}$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$

$B = \frac{F_0}{2m\omega_0}$

$$\underbrace{\boxed{x_p(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}}$$

sats $x(0)=0$, $x'(0)=0$, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class.

$$x(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

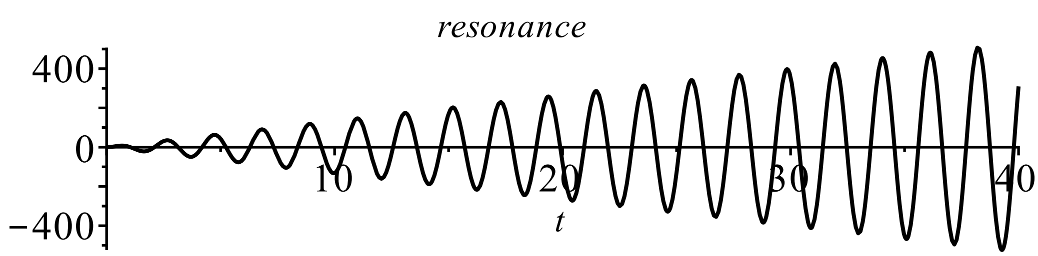
Exercise 3a) Solve the IVP

$$x'' + 9x = 80 \cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

3b) Compare the solution graph below with the beating graph in exercise 2.

> $plot\left(\frac{40}{3} t \cdot \sin(3 \cdot t), t = 0..40, color = black, title = \text{`resonance`}\right);$



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