Math 2250-004
Wed Mar 22

covering this on Tuesday

Wed: (3) finish leg parts of today's notes
(4) introduce 5.6: forced oscillations
(5) quiz

Wednesday:
(1) • Please pick up your graded work packets!!
(2) • My office hours today are canceled.

Section 5.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems).

y_P: any single particular sol'n.
y_H: general homogeneous soltns.

There are two methods we will use:

• The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 5.5.

• The method of variation of parameters is more general, and yields an integral formula for a particular solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f. It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$. We'll study the case $n = 2$ only.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in the notes.

Exercise 1) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3 .$$

Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10, b_2 = 3$.

$$L(d_1 x + d_2) = b_1 x + b_2 \overset{\text{want}}{=} 10x + 3$$

$$\begin{aligned} -5\,(y_P &= d_1 x + d_2) \\ +4\,(y_P' &= d_1) \\ +1\,(y_P'' &= 0) \\ \hline L(y_P) &= x(-5d_1) \\ &\quad + 1(-5d_2 + 4d_1) \end{aligned} \quad \overset{\text{want}}{=} \quad \begin{aligned} &10x \\ &+ 3 \cdot 1 \end{aligned}$$

match coeffs of x: $-5d_1 = 10$
" " " 1: $-5d_2 + 4d_1 = 3$

$\Rightarrow d_1 = -2$; $-5d_2 - 8 = 3$
$-5d_2 = 11$
$d_2 = -\frac{11}{5}$

$$\boxed{y_P(x) = -2x - \frac{11}{5}}$$

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4y' - 5y = 10x + 3$$

$y = y_P + y_H$

$\uparrow$

$y'' + 4y' - 5y = 0$

$p(r) = r^2 + 4r - 5$

$= (r+5)(r-1)$

roots $r = -5, 1$.

$y = y_P + y_H$

$$\boxed{y = -2x - \frac{11}{5} + c_1 e^{-5x} + c_2 e^{x}}$$

$L(y) = f$

$\left.\begin{array}{l} L(y_P) = f \\ L(y_H) = 0 \end{array}\right\} \Rightarrow L(y_P + y_H)$

$= L(y_P) + L(y_H)$

$= f + 0$

$= f$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = \boxed{14 e^{2x}}$$

Hint: try $y_P = d\,e^{2x}$ because L transforms functions of that form into ones of the form $b\,e^{2x}$, i.e. $L\left(d\,e^{2x}\right) = b\,e^{2x}$. "$d$" is your "undetermined coefficient" for $b = 14$.

$-5\,[y_P = d e^{2x}]$

$+4\,[y_P' = 2d e^{2x}]$

$1\,[y_P'' = 4d e^{2x}]$

$L(y_P) = d e^{2x}(-5 + 8 + 4)$

$= 7 d e^{2x}$

$\overset{\text{want}}{=} 14 e^{2x}$

$7d = 14$

$d = 2$

$$\boxed{y_P(x) = 2e^{2x}}$$

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x} - 20x - 6 .$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4y' - 5y = 14e^{2x} - 20x - 6$$
$$y(0) = 4$$
$$y'(0) = -4 .$$

4c) Check your answer with technology.

$L(2e^{2x}) = 14e^{2x}$

$L(-2x - \frac{11}{5}) = 10x + 3$

4a) $L(2e^{2x} - 2(-2x - \frac{11}{5}))$

$= L(2e^{2x}) - 2L(-2x - \frac{11}{5})$

$= 14e^{2x} - 2(10x + 3)$

$= 14e^{2x} - 20x - 6.$

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> with(DEtools):
> dsolve({y''(x) + 4·y'(x) − 5·y(x) = 14·e^(2·x) − 20·x − 6, y(0) = 4, y'(0) = −4});
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$$y(x) = \underbrace{\frac{8}{5}}_{c_2} e^{-5x} \underbrace{- 4}_{c_1} e^{x} + \underbrace{2e^{2x} + 4x + \frac{22}{5}}_{y_P} \quad (7)$$

4a) cont'd.

$y = y_P + y_H$

$$\boxed{y = 2e^{2x} + 4x + \frac{22}{5} + c_1 e^{x} + c_2 e^{-5x}}$$

$y(0) = 4 = 2 + \frac{22}{5} + c_1 + c_2$

$y'(0) = -4 = 4 + 4 + c_1 - 5c_2$

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = \boxed{2\cos(3x)}.$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \to V$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2\, x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d\, e^{2x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why $V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?

$V = span\{\cos 3x, \sin 3x\}$ $\quad L: V \to V$

$y_p = d_1 \cos 3x + d_2 \sin 3x$

pick up here Wed

$$\begin{aligned}
-5\,[\; y_p &= d_1 \cos 3x + d_2 \sin 3x \;]\\
+4\,[\; y' &= -3d_1 \sin 3x + 3d_2 \cos 3x \;]\\
1\,[\; y'' &= -9d_1 \cos 3x - 9 d_2 \sin 3x \;]
\end{aligned}$$

$$L(y) = \cos 3x\,(-5d_1 + 12d_2 - 9d_1) + \sin 3x\,(-5d_2 - 12d_1 - 9d_2)$$

$$\stackrel{\text{want}}{=} (\cos 3x)(2) + (\sin 3x)(0)$$

sys for d_1, d_2:

$$\begin{aligned}
-14d_1 + 12d_2 &= 2\\
-12d_1 - 14d_2 &= 0
\end{aligned}$$

$$\begin{aligned}
-7d_1 + 6d_2 &= 1\\
6d_1 + 7d_2 &= 0
\end{aligned}$$

$$\begin{bmatrix} -7 & 6 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{-85}\begin{bmatrix} 7 & -6 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{85} \\ +\frac{6}{85} \end{bmatrix}$$

wow!

$$\boxed{y_p(x) = -\frac{7}{85}\cos 3x + \frac{6}{85}\sin 3x}$$

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> with(DEtools) :
  dsolve(y"(x) + 4·y'(x) − 5·y(x) = 2·cos(3·x), y(x));
```

$$y(x) = \underbrace{e^{-5x}\,\_C2 + e^{x}\,\_C1}_{y_H} \underbrace{- \frac{7}{85}\cos(3x) + \frac{6}{85}\sin(3x)}_{\text{the } y_p \text{ we found.}} \qquad \textbf{(8)}$$

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All of the previous exercises rely on:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that

(i) $L : V \to V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

why: We already know this fact is true for matrix transformations $L(\underline{\boldsymbol{x}}) = A_{n \times n}\underline{\boldsymbol{x}}$ with $L : \mathbb{R}^n \to \mathbb{R}^n$ (because if the only homogeneous solution is $\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ then A reduces to the identity, so also each matrix equation $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution $\underline{\boldsymbol{x}}$. The theorem above is a generalization of that fact to general linear transformations. There is an "appendix" explaining this at the end of today's notes, for students who'd like to understand the details.

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

(assuming the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

6a) $L(y) = x^3 + 6x - 5$

$L : V \to V$
$f \in V$
$y_P = d_1 x^3 + d_2 x^2 + d_3 x + d_4 \cdot 1$
i.e. $V = \mathrm{span}\{x^3, x^2, x, 1\}$

6b) $L(y) = 4e^{2x}\sin(3x)$

$V = \mathrm{span}\{e^{2x}\sin 3x,\ e^{2x}\cos 3x\}$

$y_P = d_1 e^{2x}\sin 3x + d_2 e^{2x}\cos 3x$
$y_P' = b_1 e^{2x}\sin 3x + b_2 e^{2x}\cos 3x$
etc!

6c) $L(y) = x\cos(2x)$

$V = \mathrm{span}\{x\cos 2x,\ \cos 2x,\ x\sin 2x,\ \sin 2x\}$
$y_P = d_1 x\cos 2x + d_2 \cos 2x + d_3 x\sin 2x + d_4 \sin 2x$

```
> dsolve(y''(x) + 4·y'(x) − 5·y(x) = x·cos(2·x), y(x));
```

$$y(x) = \underbrace{e^{-5x}\,\_C2 + e^{x}\,\_C1}_{y_H} + \frac{1}{21025}\left(-1305\,x + 508\right)\cos(2\,x) + \frac{1}{21025}\left(1160\,x + 644\right)\sin(2\,x) \qquad (9)$$

e.g. $d_3 = \frac{1160}{21025}$

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>
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BUT LOOK OUT

Exercise 7a) Find a particular solution to

$$y'' + 4y' - 5y = 4e^x.$$

Hint: since $\boxed{y_H = c_1 e^x + c_2 e^{-5x}}$, a guess of $y_P = a e^x$ will not work (and $span\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\,x\,e^x$$

and factor $L = D^2 + 4D - 5I = [D + 5I] \circ [D - I]$.

$p(r) = r^2 + 4r - 5$
$= (r+5)(r-1)$
roots $r = -5, +1$

$L(y) = [D^2 + 4D - 5I]y$
$= y'' + 4y' - 5y$

let's try $y_p = de^x$
$L(y_p) = 0$

$[D - aI]\, g(x)e^{ax} = g'(x)e^{ax}$
$g'e^{ax} + \cancel{gae^{ax}} - \cancel{age^{ax}}$

$y_p = dxe^x$
$L(y_p) = [D+5I]\circ\underbrace{[D-I]\,dxe^x}$
$= [D+5I]\,de^x$ $(a=1)$
$= de^x + 5de^x$
$= 6de^x \overset{\text{want}}{=} 4e^x$
$d = 2/3$

$y_p = \frac{2}{3}xe^x$

OR

$-5\,[y_p = dxe^x]$
$+4\,[y_p' = de^x + dxe^x]$
$y_p'' = de^x + de^x + dxe^x$
$1\,[y_p'' = 2de^x + dxe^x]$

$L(y_p) = xe^x(\cancel{-5d + 4d + d}) \to 0$
$e^x(0 + 4d + 2d)$
$= 6de^x = 4e^x$
$6d = 4$
$d = \frac{2}{3}$
$y_p = \frac{2}{3}xe^x$ ✓

7b) check work with technology

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> with(DEtools):
> dsolve(y''(x) + 4·y'(x) − 5·y(x) = 4·e^x, y(x));
```

$$y(x) = e^{-5x}\,\_C2 + e^x\,\_C1 + \frac{2}{3}x\,e^x \qquad (10)$$

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>
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A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in lab you're working on this week) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in pages 341-346 of the text, and the table on page 346, reproduced here.

Method of undetermined coefficients (extended case): If L has a factor $(D - r I)^s$ and $e^{r x}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s , as in Exercise 7. (If you understood the homework problem last week about factoring L into composition of terms like $\boxed{(D - r I)^s}$, then you have an inkling of why this recipe works. If you didn't understand that last week problem, there's another one this week so you get a second chance. :-)) You may also need to use superposition, as in Exercise 4, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	multiply by y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega x) + b_2 \sin(\omega x)$	$x^s\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = \pm i\omega$
$e^{a x}\left(b_1 \cos(\omega x) + b_2 \sin(\omega x)\right)$	$x^s e^{a x}\left(c_1 \cos(\omega x) + c_2 \sin(\omega x)\right)$	$r = a \pm i\omega$
$b_0 e^{a x}$	$x^s c_0 e^{a x}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right) e^{a x}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right) e^{a x}$	$r = a$

e.g. (exercise 7) $y'' + 4y' - 5y = 4e^x$

• multi by x^s so that no terms in guess solve homog DE

Math 2250-004
Friday Mar 24

tomorrow Thurs in labs:
$y(x) \longrightarrow x(t)$

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on Monday using these notes.
Overview for solutions $x(t)$ to

$$\boxed{m\,x'' + c\,x' + k\,x = F_0\cos(\omega\,t)}$$

using section 5.5 undetermined coefficients algorithms.

- $\boxed{\text{undamped } (c = 0):}$
  In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}x = 0$$
$$x'' + \omega_0^2\,x = 0$$

$$m x'' + k x = F_0 \cos \omega t$$
$$x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t$$
$$x'' + \omega_0^2 x = \frac{F}{m} \cos \omega t$$

which has simple harmonic motion solutions $x_H(t) = C\cos(\omega_0 t - \alpha)$. So for the non-homogeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

$x_p = A\cos\omega t + \cancel{B\sin\omega t}$

- $\omega \neq \omega_0 := \sqrt{\dfrac{k}{m}} \Rightarrow x_P = A\cos(\omega\,t)$ because only even derivatives, we don't need $\sin(\omega\,t)$ terms !!

$$\boxed{\Rightarrow x = x_P + x_H = A\cos(\omega\,t) + C_0\cos(\omega_0 t - \alpha_0)}$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!
- $\omega = \omega_0$ $\quad\Rightarrow \boxed{x_P = t\big(A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\big)}$

$$\Rightarrow x = x_P + x_H = C\,t\cos(\omega_0\,t - \alpha) + C_0\cos(\omega_0 t - \alpha_0)\,.$$

("pure" resonance!)

$$x'' + \omega_0^2 x = \frac{F}{m} \cos \omega_0 t$$

pure reson.

....

$$\boxed{m\,x'' + c\,x' + k\,x = F_0\cos(\omega\,t)}$$

- $\boxed{\text{damped } (c > 0):}$ in all cases $x_P = A\cos(\omega\,t) + B\sin(\omega\,t) = C\cos(\omega\,t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm\,i\,\omega$ when $c > 0$).
  - underdamped: $\quad x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}C_1\cos(\omega_1 t - \alpha_1)\,.$
  - critically-damped: $x = x_P + x_H = C\cos(\omega\,t - \alpha) + e^{-p\,t}(c_1 t + c_2)\,.$
  - over-damped: $\quad x = x_P + x_H = C\cos(\omega\,t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}\,.$

$x_H(t) \longrightarrow 0$ exp. $\qquad x_H(t) = x_{tr}(t)$

steady periodic part of the soln $\qquad$ transient part.

$x_{sp}(t) = x_p(t)$

take-home quiz
hand in at labs tomorrow