Exercise 3) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:

3a)

double root (neg.)

$\boxed{\text{critically damped}}$

$$x'' + 6x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 6r + 9$
$= (r+3)^2 = 0 \quad r = -3$
$x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

> with(DEtools) :

> $dsolve\left(\left\{x''(t) + 6\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = e^{-3t} + \frac{9}{2} e^{-3t} t \tag{1}$$

Wed, pick up here.

>

3b)

overdamped $(r_1, r_2 < 0)$

$$x'' + 10x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 10r + 9$
$= (r+9)(r+1)$
$r = -9, -1$
$x(t) = c_1 e^{-t} + c_2 e^{-9t}$

> $dsolve\left(\left\{x''(t) + 10\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = \frac{21}{16} e^{-t} - \frac{5}{16} e^{-9t} \tag{2}$$

>

$\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$

$\omega_0^2 = 9, \ \omega_0 = 3$

b

c

3c)

underdamped

$r = -a \pm bi$

$$x'' + 2x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 2r + 9 = 0$
$r = \frac{-2 \pm \sqrt{4 - 36}}{2}$
$= -1 \pm \frac{\sqrt{-32}}{2}$
$= -1 \pm 2\sqrt{2}\, i$

or $(r+1)^2 + 8 = 0$
$(r+1)^2 = -8$
$r + 1 = \pm i\sqrt{8}$
$= \pm i\, 2\sqrt{2}$
$r = -1 \pm 2\sqrt{2}\, i$

> $dsolve\left(\left\{x''(t) + 2\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = \frac{5}{8}\sqrt{2}\, e^{-t} \sin\left(2\sqrt{2}\, t\right) + e^{-t} \cos\left(2\sqrt{2}\, t\right) \tag{3}$$

>

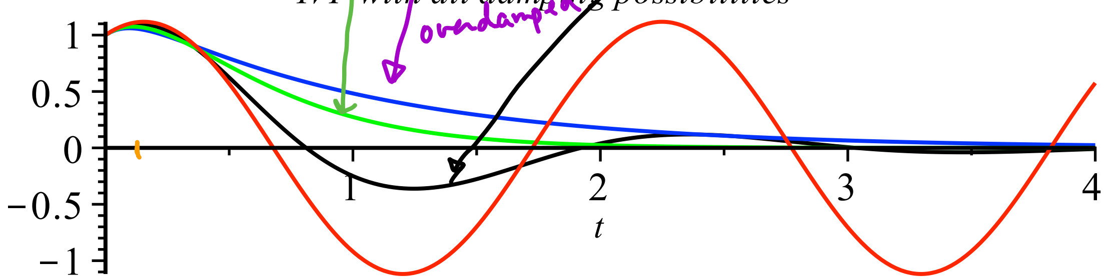
$x(t) = c_1 e^{-t} \cos(2\sqrt{2}\, t) + c_2 e^{-t} \sin(2\sqrt{2}\, t)$

$\omega_1 = 2\sqrt{2} \approx 2.8$

ω_0 for no damping $= 3$

```
> with(plots) :
> plot0 := plot(cos(3·t) + 1/2·sin(3·t), t = 0..4, color = red) :
  plot1a := plot(exp(−3·t)·(1 + 9/2·t), t = 0..4, color = green) :
  plot1b := plot(21/16·exp(−t) − 5/16·exp(−9·t), t = 0..4, color = blue) :
  plot1c := plot(5/8·√2 e^(−t)·sin(2√2·t) + e^(−t)·cos(2√2·t), t = 0..4, color = black) :
  display({plot0, plot1a, plot1b, plot1c}, title = `IVP with all damping possibilities`);
```

IVP with all damping possibilities



exercise 2 undamped.

finish

5.4 unforced mechanical "vibrations"

$mx'' + cx' + kx = 0$

5.5 on Wed notes. Find $y_p(x)$ to solve $L(y_p) = f$ (non-homog)

Math 2250-004
Week 10, March 20-24: 5.4-5.6

Mon Mar 20 Work through Fri Mar 10 notes on section 5.4: unforced mass-spring systems.

general soltn $= y_P + y_H$

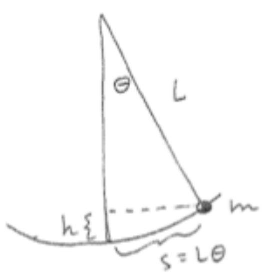
Wed: 5.5 - 5.6 $mx'' + cx' + kx = f(t)$

Tues Mar 21 If I'm able to obtain them from Physics, we'll do pendulum and mass-spring experiments :-). Then begin section 5.5 on finding particular solutions to inhomogeneous linear DE's. It's possible I won't have the experiments on Tuesday and that we'll move directly into section 5.5

Experiments postponed - we'll work on Wed notes! (after a few words about Monday's)

Experiment discussion: Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \dfrac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



$\theta = \theta(t)$

conservative system KE + PE = const.

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t)$ $\qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\frac{1}{2}m L^2(\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$

D_t: $mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$\underbrace{mL\theta'}_{\neq 0 \text{ except at isolated times}}(L\theta'' + g\sin\theta) \equiv 0$

~ deduce eqtn of motion is $\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

linearize

$\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

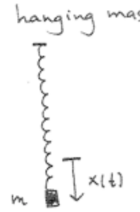
$\theta(t) = C\cos(\omega_0 t - \alpha)$

non-linear DE

but $\sin\theta = \theta - \frac{\theta^3}{3!} + \dots$

$\sin\theta \approx \theta$ θ small is excellent approx (alternating series test)

2) hanging mass-spring:



$mx'' = -kx$

$mx'' + kx = 0$

$\boxed{x'' + \frac{k}{m}x = 0}$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

covering this on Tuesday ☺

Math 2250-004
Wed Mar 22

Section 5.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems).

y_P: any single particular sol'n. y_H: general homogeneous solt'ns.

There are two methods we will use:

- The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 5.5.

- The method of variation of parameters is more general, and yields an integral formula for a particular solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f. It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$. We'll study the case $n = 2$ only.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in the notes.

Exercise 1) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3.$$

Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10$, $b_2 = 3$.

$$L(d_1 x + d_2) = b_1 x + b_2 \overset{\text{want}}{=} 10x + 3$$

$$
\begin{aligned}
-5(\ y_P &= d_1 x + d_2\ ) \\
+4(\ y_P' &= d_1\ ) \\
+1(\ y_P'' &= 0\ ) \\
\hline
L(y_P) &= x(-5d_1) + 1(-5d_2 + 4d_1) \overset{\text{want}}{=} 10x + 3\cdot 1
\end{aligned}
$$

match coeffs of x : $-5d_1 = 10$

" " " 1 : $-5d_2 + 4d_1 = 3$

$\Rightarrow d_1 = -2$; $-5d_2 - 8 = 3$

$-5d_2 = 11$

$d_2 = -\frac{11}{5}$

$$\boxed{y_P(x) = -2x - \frac{11}{5}}$$

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4y' - 5y = 10x + 3$$

$y = y_P + y_H$

$\uparrow$

$y'' + 4y' - 5y = 0$

$p(r) = r^2 + 4r - 5$

$= (r+5)(r-1)$

roots $r = -5, 1$.

$y = y_p + y_H$

$$\boxed{y = -2x - \frac{11}{5} + c_1 e^{-5x} + c_2 e^{x}}$$

$L(y) = f$

$\left.\begin{array}{l} L(y_p) = f \\ L(y_H) = 0 \end{array}\right\} \Rightarrow L(y_p + y_H)$

$= L(y_p) + L(y_H)$

$= f + 0$

$= f$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = \boxed{14e^{2x}}$$

Hint: try $y_P = d\,e^{2x}$ because L transforms functions of that form into ones of the form $b\,e^{2x}$, i.e. $L\left(d\,e^{2x}\right) = b\,e^{2x}$. "$d$" is your "undetermined coefficient" for $b = 14$.

$-5\,[y_p = de^{2x}]$

$+4\,[y_p' = 2de^{2x}]$

$1\,[y_p'' = 4de^{2x}]$

$L(y_p) = de^{2x}(-5 + 8 + 4)$

$= 7de^{2x}$

$\overset{\text{want}}{=} 14e^{2x}$

$7d = 14$

$d = 2$

$$\boxed{y_p(x) = 2e^{2x}}$$

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x} - 20x - 6 .$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4y' - 5y = 14e^{2x} - 20x - 6$$
$$\left\{\begin{array}{l} y(0) = 4 \\ y'(0) = -4 . \end{array}\right.$$

4c) Check your answer with technology.

$L(2e^{2x}) = 14e^{2x}$

$L(-2x - \frac{11}{5}) = 10x + 3$

4a) $L\left(2e^{2x} - 2(-2x - \frac{11}{5})\right)$
$= L(2e^{2x}) - 2L(-2x - \frac{11}{5})$
$= 14e^{2x} - 2(10x + 3)$
$= 14e^{2x} - 20x - 6.$

```
> with(DEtools):
> dsolve({y''(x) + 4·y'(x) − 5·y(x) = 14·e^(2·x) − 20·x − 6, y(0) = 4, y'(0) = −4});
```

$$y(x) = \underbrace{\frac{8}{5}}_{c_2} e^{-5x} \underbrace{-4}_{c_1} e^{x} + \underbrace{2e^{2x} + 4x + \frac{22}{5}}_{y_P} \qquad (7)$$

```
>
```

4a) cont'd.

$y = y_P + y_H$

$$\boxed{y = 2e^{2x} + 4x + \frac{22}{5} + c_1 e^{x} + c_2 e^{-5x}}$$

$y(0) = 4 = 2 + \frac{22}{5} + c_1 + c_2$

$y'(0) = -4 = 4 + 4 + c_1 - 5c_2$

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2\cos(3x) .$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \rightarrow V$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d\, e^{2x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why

$V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?

$V = span\{\cos 3x, \sin 3x\}$ $\quad L: V \to V$

$y_P = d_1 \cos 3x + d_2 \sin 3x$

```
> with(DEtools) :
  dsolve(y"(x) + 4·y'(x) − 5·y(x) = 2·cos(3·x), y(x));
```

$$y(x) = e^{-5x}\,\_C2 + e^{x}\,\_C1 - \frac{7}{85}\cos(3x) + \frac{6}{85}\sin(3x) \qquad \textbf{(8)}$$

```
>
```