Friday : Wednesday notes
at end of class, introduce § 5.4
Monday 3/20 start here

Friday Mar 10

$\boxed{5.4:}$ Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

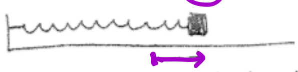
In this section we study the differential equation below for functions $x(t)$:

$$m\,x'' + c\,x' + k\,x = 0\,.$$

equil loc. →

x(t)

model



(m)

x(t) displacement from equilibrium.

equilibrium location

possible external forcing function

Newton's 2^{nd} Law: $m\,x''(t) = \text{net forces}$

$$= \underbrace{-c\,x'}\ \underbrace{-\,k\,x}\ + \underbrace{F(t)}$$

linear drag, proportional to velocity

Hooke's Law

force from compression or stretching of spring, relative to equilibrium $x=0$

§5.3 $y(x)$ §5.4 $x(t)$

$m\,x'' + c\,x' + k\,x = F(t)$

§5.4 $F = 0$

In section 5.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero. Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t)$, $x'(t)$ are sufficiently small. k is called the <u>Hooke's constant</u>, and c is called the <u>damping coefficient</u>.

§5.4 $m\,x'' + c\,x' + k\,x = 0$

$x(t)$

$\boxed{mx'' + cx' + kx = 0}$

This is a constant coefficient linear homogeneous DE, so we try $x(t) = \boxed{e^{rt}}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{rt}\left(m\,r^2 + c\,r + k\right) = e^{rt}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ processes...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$
$$x'' + \frac{k}{m}\,x = 0\,.$$
$$p(r) = r^2 + \frac{k}{m} = 0$$

$p(r) = mr^2 + k = 0$

has roots

$$r^2 = -\frac{k}{m} \qquad \text{i.e.} \qquad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

$r = a \pm bi$ $\quad e^{rt}$ complex soln.

$x(t) = e^{at}\cos bt$

$y(t) = e^{at}\sin bt$

$\boxed{§5.3}$

$a = 0$

$b = \sqrt{\frac{k}{m}}$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

radians/time

We write $\boxed{\sqrt{\frac{k}{m}} := \omega_0}$ and call ω_0 the <u>natural angular frequency</u>. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\,x = 0$$

is

$$x(t) = A\cos(\omega_0\,t) + B\sin(\omega_0\,t)\,.$$

This motion is called <u>simple harmonic motion.</u> The reason for this is that $x(t)$ can be rewritten as

$$x(t) = C\cos(\omega_0\,t - \alpha) = C\cos(\omega_0(t - \delta))$$

in terms of an <u>amplitude</u> $C > 0$ and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ).

$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\sin(a+b) = \cos a \sin b + \sin a \cos b$$

this contains all of trigonometry!

To see why functions of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

are equal (for appropriate choices of constants) to ones of the form

$$x(t) = C\cos(\omega_0 t - \alpha)$$

we use the very important the addition angle trigonometry identities, in this case the addition angle for *cosine* : Consider the possible equality of functions

$$\boxed{A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha)}\,.$$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\boxed{\text{Exercise 1)}}$ Use the addition angle formula $\boxed{\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)}$ to show that the two functions above are equal provided

$= \cos a \cos(-b) - \sin a \sin(-b)$

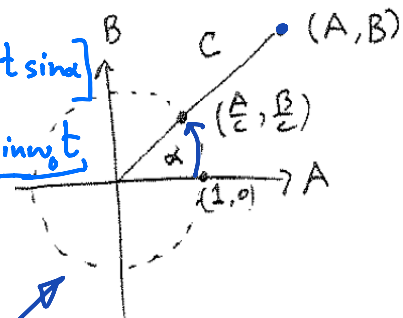
$$A = C\cos\alpha$$
$$B = C\sin\alpha\ .$$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:

$$A\cos\omega_0 t + B\sin\omega_0 t \stackrel{?}{=} C\cos(\omega_0 t - \alpha)$$
$$= C\left[\cos\omega_0 t\cos\alpha + \sin\omega_0 t\sin\alpha\right]$$
$$= C\cos\alpha\cos\omega_0 t + C\sin\alpha\sin\omega_0 t$$



$$\Rightarrow \boxed{\begin{aligned} A &= C\cos\alpha \\ B &= C\sin\alpha \end{aligned}}$$

$$\Rightarrow A^2 + B^2 = C^2\cos^2\alpha + C^2\sin^2\alpha$$
$$= C^2$$

$$\boxed{\begin{aligned} &C \Rightarrow \sqrt{A^2+B^2} \\ &\frac{A}{C} = \cos\alpha \\ &\frac{B}{C} = \sin\alpha \end{aligned}}$$

It is important to understand the behavior of the functions

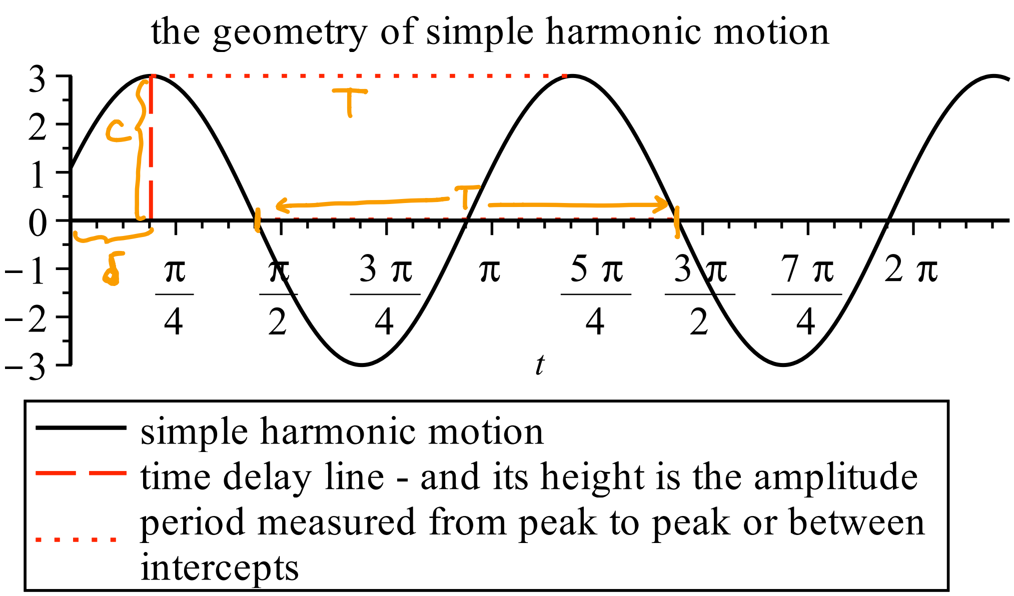
$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

$$f = \text{frequency} \;=\; \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} \;=\; \frac{2\pi}{\omega_0} \;=\; \text{time/cycle.}$$



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0 ..7, color = black) :
  plot2 := plot([.6, t, t = 0 ..3.], linestyle = dash) :
  plot3 := plot(3, t = .6 ..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

$2x'' + 18x = 0$

Exercise 2) A mass of 2 *kg* oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 *m* from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

$x(0) = 1$ $\quad x'(0) = 3/2$

2a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$\begin{cases} x'' + \boxed{9}\,x = 0 \\ x(0) = 1 \\ x'(0) = \dfrac{3}{2}. \end{cases}$$

$\omega_0^2 = 9$

$\omega_0 = 3$

2b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the commands below.

$p(r) = r^2 + 9 = 0$

$r^2 = -9$

$r = \pm 3i$ $\qquad e^{\pm 3it}$

$x_H(t) = A\cos 3t + B\sin 3t$ $\qquad x_H'(t) = -3A\sin 3t + 3B\cos 3t$

$x_H'(0) = 0 + 3B$

IVP

$x(0) = \boxed{1 = A}$

$x'(0) = 3/2 = 3B \Rightarrow \boxed{B = \frac{1}{2}}$

$x(t) = \cos 3t + \frac{1}{2}\sin 3t$

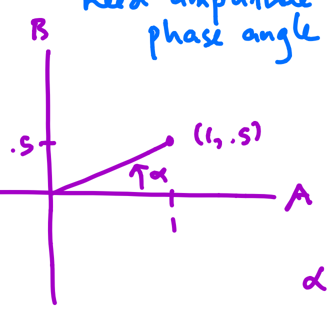
$= C\cos(3t - \alpha)$ $\qquad$ need amplitude C, phase angle α

$C = \sqrt{A^2 + B^2} = \sqrt{1.25}$

≈ 1.12

$x(t) \cong 1.12\cos(3t - .46)$

$\cong 1.12\cos(3(t - .151))$



$\cos\alpha = \frac{1}{\sqrt{1.25}}$

$\sin\alpha = \frac{.5}{\sqrt{1.25}}$

$\tan\alpha = \frac{.5}{1} = .5$

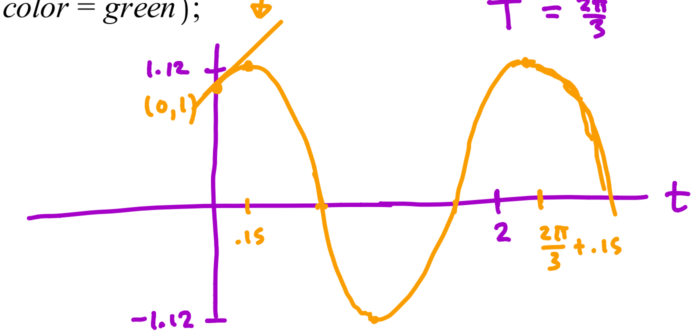
$\alpha = .46$ rad

```
> unassign(`x`);
> with(plots):
  with(DEtools):
> dsolve({x''(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
> plot(rhs(%), t = 0..5, color = green);
>
```

$\omega_0 = 3$ rad/sec.

$f = \frac{3}{2\pi}$ cycles/sec

$T = \frac{2\pi}{3}$ sec/cycle



• Then, if time, discuss the possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

damping coeff.

$m, c, k > 0$

Case 2: damping

$$m x'' + c x' + k x = 0$$

$$x'' + \frac{c}{m} x' + \frac{k}{m} x = 0$$

$x(t) = e^{rt}$

$L(x) = e^{rt}\underbrace{(mr^2 + cr + k)}_{p(r)} = 0$

$\frac{p(r)}{m}$

$r^2 + \frac{c}{m} r + \frac{k}{m} = 0$

rewrite as

$$x'' + 2 p x' + \omega_0^2 x = 0.$$

$\left(p = \frac{c}{2m}, \omega_0^2 = \frac{k}{m}\right)$. The characteristic polynomial is

$$\boxed{r^2 + 2 p r + \omega_0^2 = 0}$$

which has roots

$$r = \frac{-2 p \pm \sqrt{4 p^2 - 4\omega_0^2}}{2} \boxed{= -p \pm \sqrt{p^2 - \omega_0^2}\,.}$$

2a) $(p^2 > \omega_0^2$, or $c^2 > 4 m k)$. $\boxed{\text{overdamped.}}$ In this case we have two negative real roots

$$r_1 < r_2 < 0$$

$r_1 = -p - \sqrt{p^2 - \omega_0^2}$

$r_2 = -p + \sqrt{p^2 - \omega_0^2} < 0$

and

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{r_2 t}\left(c_1 \underbrace{e^{(r_1 - r_2) t}}_{} + c_2\right).$$

• solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

>0

has at most one real root $(t>0)$

2b) ($p^2 = \omega_0^2$, or $c^2 = 4\,m\,k$) $\boxed{\text{critically damped}}$. Double real root $r_1 = r_2 = -p = -\frac{c}{2\,m}$.

$$\boxed{x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right)}\,. \qquad e^{-pt},\ t e^{-pt}$$

- solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

2c) ($p^2 < \omega_0^2$, or $c^2 < 4\,m\,k$) $\boxed{\text{underdamped}}$. Complex roots

$$\sqrt{p^2-\omega_0^2} = \sqrt{-(\omega_0^2-p^2)} = i\sqrt{\omega_0^2-p^2} = i\,\omega_1$$

$$r = -p \pm \sqrt{\underbrace{p^2 - \omega_0^2}_{<0}} = -p \pm i\,\omega_1$$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$$e^{-pt}\cos\omega_1 t,\ e^{-pt}\sin\omega_1 t$$

$$x(t) = e^{-p\,t}\left(A\cos\left(\omega_1 t\right) + B\sin\left(\omega_1\,t\right)\right) = e^{-p\,t}C\cos\left(\omega_1 t - \alpha_1\right).$$

- solution decays exponentially to zero, but oscillates infinitely often, with exponentially decaying pseudo-amplitude $e^{-p\,t}C$ and pseudo-angular frequency ω_1, and pseudo-phase angle α_1 .

Exercise 3) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:

3a)

$$x'' + 6x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$\boxed{\text{critically damped}}$

$p(r) = r^2 + 6r + 9$
$= (r+3)^2$
$x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

> $with(DEtools)$:

> $dsolve\left(\left\{x''(t) + 6 \cdot x'(t) + 9 \cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = e^{-3t} + \frac{9}{2} e^{-3t} t \tag{1}$$

>

3b)

$$x'' + 10x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

> $dsolve\left(\left\{x''(t) + 10 \cdot x'(t) + 9 \cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = \frac{21}{16} e^{-t} - \frac{5}{16} e^{-9t} \tag{2}$$

>

3c)

$$x'' + 2x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

> $dsolve\left(\left\{x''(t) + 2 \cdot x'(t) + 9 \cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right);$

$$x(t) = \frac{5}{8} \sqrt{2}\, e^{-t} \sin\left(2\sqrt{2}\, t\right) + e^{-t} \cos\left(2\sqrt{2}\, t\right) \tag{3}$$

>

> $with(plots):$

> $plot0 := plot\left(\cos(3\cdot t) + \frac{1}{2}\cdot\sin(3\cdot t),\ t = 0\,..4,\ color = red\right):$

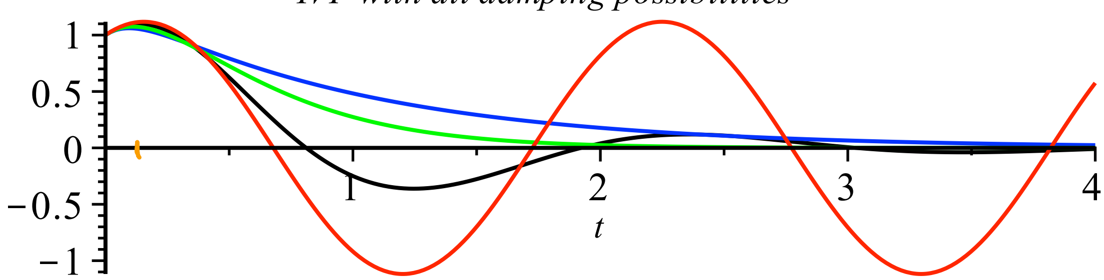
$plot1a := plot\left(\exp(-3\cdot t)\cdot\left(1 + \frac{9}{2}\cdot t\right),\ t = 0\,..4,\ color = green\right):$

$plot1b := plot\left(\frac{21}{16}\cdot\exp(-t) - \frac{5}{16}\cdot\exp(-9\cdot t),\ t = 0\,..4,\ color = blue\right):$

$plot1c := plot\left(\frac{5}{8}\cdot\sqrt{2}\,\mathrm{e}^{-t}\cdot\sin\left(2\sqrt{2}\cdot t\right) + \mathrm{e}^{-t}\cdot\cos\left(2\sqrt{2}\cdot t\right),\ t = 0\,..4,\ color = black\right):$

$display(\{plot0, plot1a, plot1b, plot1c\},\ title = \text{`IVP with all damping possibilities`});$

IVP with all damping possibilities



exercise 2