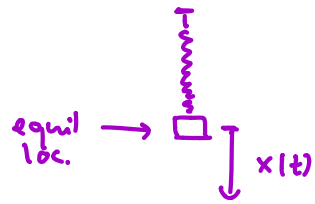
Friday : Wednesday notes
at end of class, introduce § 5.4
Monday 3/20 start here

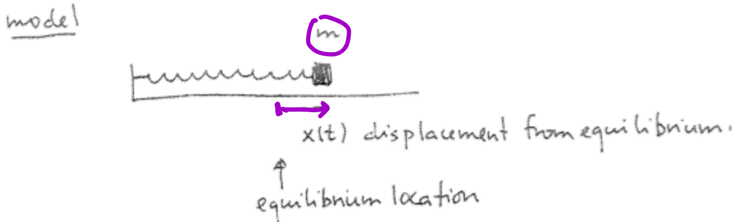
Friday Mar 10

$\boxed{5.4:}$ Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$:

$$m\,x'' + c\,x' + k\,x = 0\,.$$





Newton's 2^{nd} Law: $m\,x''(t) = $ net forces

$$= \underbrace{-c\,x'}_{\text{linear drag, proportional to velocity}} \; \underbrace{-\,k\,x}_{\text{Hooke's Law}} \; + \underbrace{F(t)}_{\text{possible external forcing function}}$$

force from compression or stretching of spring, relative to equilibrium $x = 0$

$m x'' + c x' + k x = F(t)$

§5.3 $y(x)$ §5.4 $x(t)$

§5.4 $F = 0$

In section 5.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero. Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t)$, $x'(t)$ are sufficiently small. k is called the <u>Hooke's constant</u>, and c is called the <u>damping coefficient</u>.

§5.4 $m x'' + c x' + k x = 0$

$x(t)$

$\boxed{mx'' + cx' + kx = 0}$

This is a constant coefficient linear homogeneous DE, so we try $x(t) = \boxed{e^{rt}}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{rt}\left(m\,r^2 + c\,r + k\right) = e^{rt}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ processes...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}\,x = 0\,.$$

$$p(r) = r^2 + \frac{k}{m} = 0$$

$p(r) = mr^2 + k = 0$

has roots

$$r^2 = -\frac{k}{m} \qquad \text{i.e.} \quad r = \pm\, i\sqrt{\frac{k}{m}}\,.$$

$r = a \pm bi$ $\quad e^{rt}$ complex soln.

$x(t) = e^{at}\cos bt$

$y(t) = e^{at}\sin bt$

§5.3

$a = 0$

$b = \sqrt{\frac{k}{m}}$

So the general solution is

$$x(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

radians/time

We write $\boxed{\sqrt{\frac{k}{m}} := \omega_0}$ and call ω_0 the <u>natural angular frequency</u>. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$\boxed{x'' + \omega_0^{\,2}\,x = 0}$$

is

$$x(t) = A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right).$$

This motion is called <u>simple harmonic motion.</u> The reason for this is that $x(t)$ can be rewritten as

$$x(t) = C\cos\left(\omega_0\,t - \alpha\right) = C\cos\left(\omega_0\left(t - \delta\right)\right)$$

in terms of an <u>amplitude</u> $C > 0$ and a <u>phase angle</u> α (or in terms of a <u>time delay</u> δ).

To see why functions of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

are equal (for appropriate choices of constants) to ones of the form

$$x(t) = C\cos(\omega_0 t - \alpha)$$

$\cos^2\theta + \sin^2\theta = 1$
$\cos(a+b) = \cos a\cos b - \sin a\sin b$
$\sin(a+b) = \cos a\sin b + \sin a\cos b$
this contains all of trigonometry.

we use the very important the addition angle trigonometry identities, in this case the addition angle for *cosine* : Consider the possible equality of functions

$$\boxed{A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha)}\ .$$

$\boxed{\text{Exercise 1)}}$ Use the addition angle formula $\cos(a+b) = \cos a\cos b - \sin a\sin b$

$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ $= \cos a\cos(-b) - \sin a\sin(-b)$ to show that the two functions above are equal provided

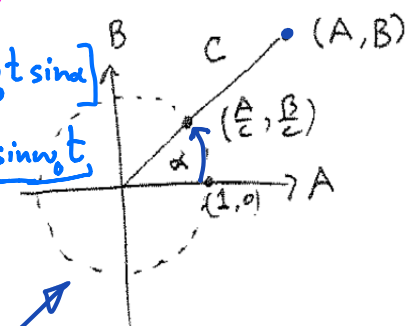
$$A = C\cos\alpha$$
$$B = C\sin\alpha\ .$$

So if C, α are given, the formulas above determine A, B. Conversely, if A, B are given then

$$C = \sqrt{A^2 + B^2}$$
$$\frac{A}{C} = \cos(\alpha),\ \frac{B}{C} = \sin(\alpha)$$

determine C, α. These correspondences are best remembered using a diagram in the $A - B$ plane:

$$\underbrace{A}\ \underbrace{\cos\omega_0 t} + \underbrace{B}\ \underbrace{\sin\omega_0 t} \overset{?}{=} C\cos(\omega_0 t - \alpha)$$
$$= C\left[\cos\omega_0 t\cos\alpha + \sin\omega_0 t\sin\alpha\right]$$
$$= \underbrace{C\cos\alpha}\ \underbrace{\cos\omega_0 t} + \underbrace{C\sin\alpha}\ \underbrace{\sin\omega_0 t}$$



$$\Rightarrow \boxed{\begin{array}{l} A = C\cos\alpha \\ B = C\sin\alpha \end{array}}$$

$$\Rightarrow A^2 + B^2 = C^2\cos^2\alpha + C^2\sin^2\alpha$$
$$= C^2$$

$$\boxed{\begin{array}{l} C \Rightarrow \sqrt{A^2+B^2} \\ \frac{A}{C} = \cos\alpha \\ \frac{B}{C} = \sin\alpha \end{array}}$$

It is important to understand the behavior of the functions

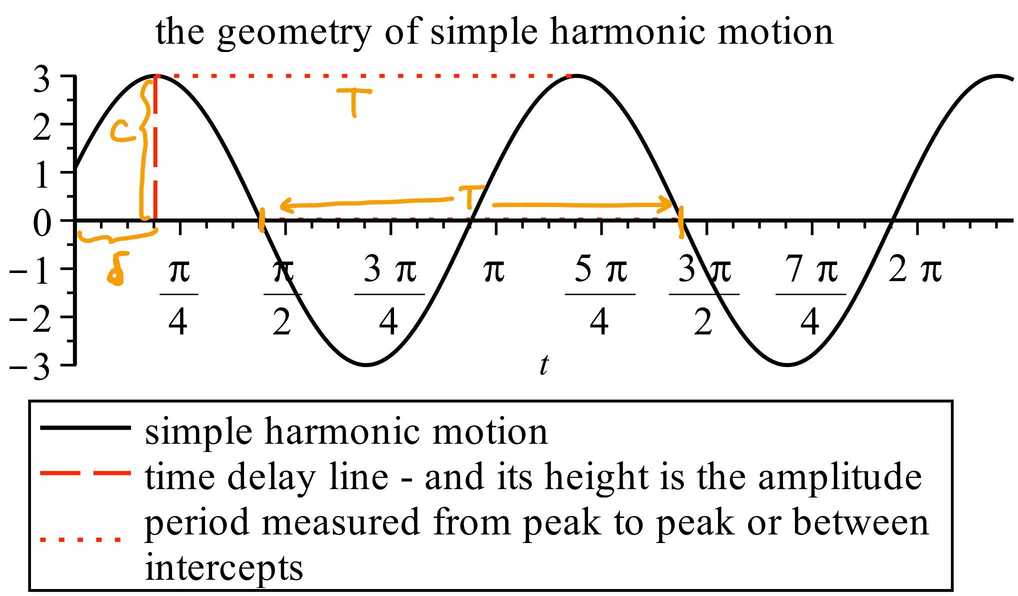
$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \alpha) = C\cos(\omega_0 (t - \delta))$$

and the standard terminology:

The <u>amplitude</u> C is the maximum absolute value of $x(t)$. The time delay δ is how much the graph of $C\cos(\omega_0 t)$ is shifted to the right in order to obtain the graph of $x(t)$. Other important data is

$$f = \text{frequency} = \frac{\omega_0}{2\pi} \quad \text{cycles/time}$$

$$T = \text{period} = \frac{2\pi}{\omega_0} = \text{time/cycle.}$$



(I made that plot above with these commands...and then added a title and a legend, from the plot options.)

```
> with(plots) :
> plot1 := plot(3·cos(2(t − .6)), t = 0..7, color = black) :
  plot2 := plot([.6, t, t = 0..3.], linestyle = dash) :
  plot3 := plot(3, t = .6..(.6) + Pi, linestyle = dot) :
  plot4 := plot(0.02, t = .6 + Pi/4 ...6 + 5·Pi/4, linestyle = dot) :
> display({plot1, plot2, plot3, plot4});
>
```

$2x'' + 18x = 0$

Exercise 2) A mass of 2 *kg* oscillates without damping on a spring with Hooke's constant $k = 18\ \frac{N}{m}$. It is initially stretched 1 *m* from equilibrium, and released with a velocity of $\frac{3}{2}\ \frac{m}{s}$.

$x(0) = 1$ $\qquad x'(0) = 3/2$

2a) Show that the mass' motion is described by $x(t)$ solving the initial value problem

$$\begin{cases} x'' + \boxed{9}\,x = 0 \\ x(0) = 1 \\ x'(0) = \frac{3}{2}. \end{cases}$$

$\omega_0^2 = 9$

$\omega_0 = 3$

2b) Solve the IVP in a, and convert $x(t)$ into amplitude-phase and amplitude-time delay form. Sketch the solution, indicating amplitude, period, and time delay. Check your work with the commands below.

$p(r) = r^2 + 9 = 0$

$r^2 = -9$

$r = \pm 3i$ $\qquad e^{\pm 3it}$

$x_H(t) = A\cos 3t + B\sin 3t$ $\qquad x_H'(t) = -3A\sin 3t + 3B\cos 3t$

$x_H'(0) = 0 + 3B$

IVP

$x(0) = \boxed{1 = A}$

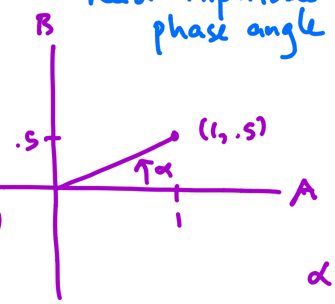
$x'(0) = 3/2 = 3B \Rightarrow \boxed{B = \frac{1}{2}}$

$x(t) = \cos 3t + \frac{1}{2}\sin 3t$

$= C\cos(3t - \alpha)$ $\qquad$ need amplitude C, phase angle α

$C = \sqrt{A^2 + B^2} = \sqrt{1.25}$

≈ 1.12



$\cos\alpha = \frac{1}{\sqrt{1.25}}$

$\sin\alpha = \frac{.5}{\sqrt{1.25}}$

$\tan\alpha = \frac{.5}{1} = .5$

$\alpha = .46$ rad

$x(t) \cong 1.12\cos(3t - .46)$

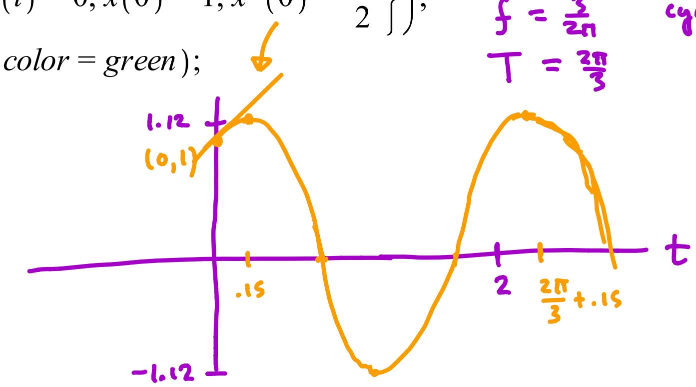
$\cong 1.12\cos(3(t - .151))$

```
> unassign(`x`);
> with(plots) :
  with(DEtools) :
> dsolve({x''(t) + 9·x(t) = 0, x(0) = 1, x'(0) = 3/2});
> plot(rhs(%), t = 0..5, color = green);
>
```

$\omega_0 = 3$ rad/sec.

$f = \frac{3}{2\pi}$ cycles/sec

$T = \frac{2\pi}{3}$ sec/cycle



• Then, if time, discuss the possibilities that arise when the damping coefficient $c > 0$. There are three cases, depending on the roots of the characteristic polynomial:

damping coeff.

$m, c, k > 0$

Case 2: damping

$$m\,x'' + c\,x' + k\,x = 0$$

$$x'' + \frac{c}{m}\,x' + \frac{k}{m}\,x = 0$$

$x(t) = e^{rt}$

$L(x) = e^{rt}\underbrace{(mr^2 + cr + k)}_{p(r)} = 0$

$\frac{p(r)}{m}$

$r^2 + \frac{c}{m}r + \frac{k}{m} = 0$

rewrite as

$$x'' + 2\,p\,x' + \omega_0^2\,x = 0.$$

$\left(p = \dfrac{c}{2\,m},\ \omega_0^2 = \dfrac{k}{m}\right)$. The characteristic polynomial is

$$\boxed{r^2 + 2\,p\,r + \omega_0^2 = 0}$$

which has roots

$$r = \frac{-2\,p \pm \sqrt{4\,p^2 - 4\omega_0^2}}{2} = \boxed{-p \pm \sqrt{p^2 - \omega_0^2}}.$$

2a) $(p^2 > \omega_0^2$, or $c^2 > 4\,m\,k)$. overdamped. In this case we have two negative real roots

$$r_1 < r_2 < 0$$

$r_1 = -p - \sqrt{p^2 - \omega_0^2}$

$r_2 = -p + \sqrt{p^2 - \omega_0^2} < 0$

and

$$x(t) = c_1 e^{r_1 t} + c_2\,e^{r_2 t} = e^{r_2 t}\left(c_1 e^{(r_1 - r_2)\,t} + c_2\right).$$

• solution converges to zero exponentially fast; solution passes through equilibrium location $x = 0$ at most once.

>0 has at most one real root $(t>0)$

$\underline{2b)}$ ($p^2 = \omega_0^2$, or $c^2 = 4\,m\,k$) $\boxed{\underline{\text{critically damped}}}$. Double real root $r_1 = r_2 = -p = -\dfrac{c}{2\,m}$.

$$\boxed{x(t) = e^{-p\,t}\left(c_1 + c_2\,t\right)} .$$

e^{-pt}, te^{-pt}

• solution converges to zero exponentially fast, passing through $x = 0$ at most once, just like in the overdamped case. The critically damped case is the transition between overdamped and underdamped:

$\underline{2c)}$ ($p^2 < \omega_0^2$, or $c^2 < 4\,m\,k$) $\boxed{\underline{\text{underdamped}}}$. Complex roots

$\sqrt{p^2-\omega_0^2} = \sqrt{-(\omega_0^2-p^2)}$

$= i\sqrt{\omega_0^2-p^2}$

$$r = -p \pm \sqrt{\underbrace{p^2 - \omega_0^2}_{<0}} = -p \pm i\,\omega_1$$

$= i\,\omega_1$

$\omega_1 < \omega_0$

with $\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$.

$e^{-pt}\cos\omega_1 t$, $e^{-pt}\sin\omega_1 t$

$$x(t) = \underline{e^{-p\,t}\left(A\cos\left(\omega_1 t\right) + B\sin\left(\omega_1 t\right)\right)} = e^{-p\,t}C\cos\left(\omega_1 t - \alpha_1\right) .$$

• solution decays exponentially to zero, <u>but</u> oscillates infinitely often, with exponentially decaying <u>pseudo-amplitude</u> $e^{-p\,t}C$ and <u>pseudo-angular frequency</u> ω_1, and <u>pseudo-phase angle</u> α_1 .

Exercise 3) Classify by finding the roots of the characteristic polynomial. Then solve for $x(t)$:

3a)

double root (neg.)

$\boxed{\text{critically damped}}$

$$x'' + 6x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 6r + 9$
$= (r+3)^2 = 0 \quad r = -3$
$x(t) = c_1 e^{-3t} + c_2 t e^{-3t}$

> with(DEtools) :

> dsolve$\left(\left\{x''(t) + 6\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right)$;

Wed, pick up here.

$$x(t) = e^{-3t} + \frac{9}{2} e^{-3t} t \qquad (1)$$

>

3b)

overdamped $(r_1, r_2 < 0)$

$$x'' + 10x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 10r + 9$
$= (r+9)(r+1)$
$r = -9, -1$
$x(t) = c_1 e^{-t} + c_2 e^{-9t}$

> dsolve$\left(\left\{x''(t) + 10\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right)$;

$$x(t) = \frac{21}{16} e^{-t} - \frac{5}{16} e^{-9t} \qquad (2)$$

>

$\omega_1 = \sqrt{\omega_0^2 - p^2} < \omega_0$

$\omega_0^2 = 9, \ \omega_0 = 3$

3c)

underdamped

$r = -a \pm bi$

$$x'' + 2x' + 9x = 0$$
$$x(0) = 1$$
$$x'(0) = \frac{3}{2}.$$

$p(r) = r^2 + 2r + 9 = 0$
$r = \frac{-2 \pm \sqrt{4 - 36}}{2}$
$= -1 \pm \frac{\sqrt{-32}}{2}$
$= -1 \pm 2\sqrt{2}\, i$

or $(r+1)^2 + 8 = 0$
$(r+1)^2 = -8$
$r + 1 = \pm i\sqrt{8}$
$= \pm i\, 2\sqrt{2}$
$r = -1 \pm 2\sqrt{2}\, i$

> dsolve$\left(\left\{x''(t) + 2\cdot x'(t) + 9\cdot x(t) = 0, x(0) = 1, x'(0) = \frac{3}{2}\right\}\right)$;

$$x(t) = \frac{5}{8}\sqrt{2}\, e^{-t} \sin\left(2\sqrt{2}\, t\right) + e^{-t} \cos\left(2\sqrt{2}\, t\right) \qquad (3)$$

>

$x(t) = c_1 e^{-t} \cos(2\sqrt{2}\, t) + c_2 e^{-t} \sin(2\sqrt{2}\, t)$

$\omega_1 = 2\sqrt{2} \approx 2.8$

ω_0 for no damping $= 3$

> *with*(*plots*) :

> $plot0 := plot\left(\cos(3\cdot t) + \frac{1}{2}\cdot\sin(3\cdot t), t = 0..4, color = red\right)$:

$plot1a := plot\left(\exp(-3\cdot t)\cdot\left(1 + \frac{9}{2}\cdot t\right), t = 0..4, color = green\right)$:

critically damped

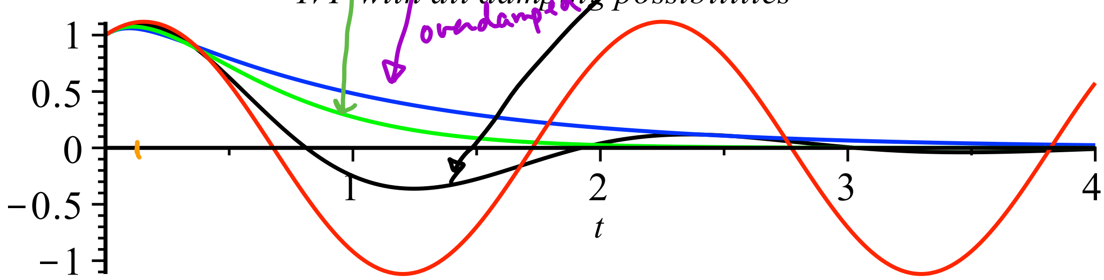
$plot1b := plot\left(\frac{21}{16}\cdot\exp(-t) - \frac{5}{16}\cdot\exp(-9\cdot t), t = 0..4, color = blue\right)$:

$plot1c := plot\left(\frac{5}{8}\cdot\sqrt{2}\,e^{-t}\cdot\sin\left(2\sqrt{2}\cdot t\right) + e^{-t}\cdot\cos\left(2\sqrt{2}\cdot t\right), t = 0..4, color = black\right)$:

display({*plot0*, *plot1a*, *plot1b*, *plot1c*}, *title* = \`*IVP with all damping possibilities*\`);

underdamped

IVP with all damping possibilities



exercise 2 undamped.

finish

5.4 unforced mechanical "vibrations"

$mx'' + cx' + kx = 0$

5.5 on Wed notes. Find $y_p(x)$ to solve

$L(y_p) = f$ (non-homog)

general soln $= y_P + y_H$

Math 2250-004
Week 10, March 20-24: 5.4-5.6

Mon Mar 20 Work through Fri Mar 10 notes on section 5.4: unforced mass-spring systems.

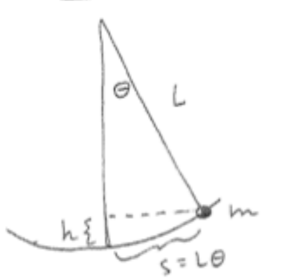
Wed: 5.5 - 5.6 $\quad mx'' + cx' + kx = f(t)$

Tues Mar 21 If I'm able to obtain them from Physics, we'll do pendulum and mass-spring experiments :-). Then begin section 5.5 on finding particular solutions to inhomogeneous linear DE's. It's possible I won't have the experiments on Tuesday and that we'll move directly into section 5.5

Experiments postponed – we'll work on Wed notes! (after a few words about Monday's)

Experiment discussion: Small oscillation pendulum motion and vertical mass-spring motion are governed by exactly the "same" differential equation that models the motion of the mass in a horizontal mass-spring configuration. The nicest derivation for the pendulum depends on conservation of mass, as indicated below. Today we will test both models with actual experiments (in the undamped cases), to see if the predicted periods $T = \dfrac{2\pi}{\omega_0}$ correspond to experimental reality.

1) pendulum



$\theta = \theta(t)$

conservative system $\quad KE + PE = \text{const.}$

$\frac{1}{2}mv^2 + mgh = \text{const}$

$s = L\theta$

$v = \frac{ds}{dt} = L\theta'(t) \qquad h = L - L\cos\theta = L(1-\cos\theta)$

so, $\quad \frac{1}{2}m L^2(\theta'(t))^2 + mgL(1-\cos(\theta(t))) \equiv \text{const}$

$D_t: \quad mL^2\theta'\theta'' + mgL(\sin\theta)\theta' \equiv 0$

$\underbrace{mL\theta'}_{\substack{\neq 0 \text{ except} \\ \text{at isolated} \\ \text{times}}}(L\theta'' + g\sin\theta) \equiv 0$

$\rightsquigarrow$ deduce eqtn of motion is $\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}$

linearize

$\boxed{\theta'' + \frac{g}{L}\theta = 0}$

$\omega_0 = \sqrt{\frac{g}{L}}$

$\theta(t) = C\cos(\omega_0 t - \alpha)$

non-linear DE

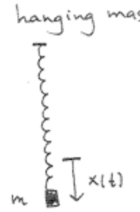
but $\sin\theta = \theta - \frac{\theta^3}{3!} + \ldots$

$\sin\theta \approx \theta \quad \theta$ small

is excellent approx

(alternating series test)

2) hanging mass-spring:



$mx'' = -kx$

$mx'' + kx = 0$

$\boxed{x'' + \frac{k}{m}x = 0}$

$\omega_0 = \sqrt{\frac{k}{m}}$

Why don't you see gravity g in this DE?

Pendulum: measurements and prediction (we'll check these numbers).

```
> restart :
  Digits := 4 :
```

```
> L := 1.53;
  g := 9.806;
```

$$\omega := \sqrt{\frac{g}{L}}\text{; \# radians per second}$$

$$f := evalf(\omega / (2 \cdot \mathrm{Pi}))\text{; \# cycles per second}$$

$$T := 1/f\text{; \# seconds per cycle}$$

$$L := 1.53$$

$$g := 9.806$$

$$\omega := 2.531629974$$

$$f := 0.4029214244$$

$$T := 2.481873486 \qquad \textbf{(1)}$$

Experiment:

Mass-spring:

compute Hooke's constant:

$$> 98.7 - 83.4;\ \text{\#displacement from extra 50g}$$

$$15.3 \qquad \textbf{(2)}$$

$$> k := \frac{.05 \cdot 9.806}{.153};\ \text{\# solve } k \cdot x = m \cdot g \text{ for } k.$$

$$k := 3.204575163 \qquad \textbf{(3)}$$

$$> m := .1;\ \text{\# mass for experiment is 100g}$$

$$\omega := \sqrt{\frac{k}{m}};\ \text{\# predicted angular frequency}$$

$$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right);\ \text{\# predicted frequency}$$

$$T := \frac{1}{f};\ \text{\# predicted period}$$

$$m := 0.1$$

$$\omega := 5.660896716$$

$$f := 0.9009596945$$

$$T := 1.109927565 \qquad \textbf{(4)}$$

Experiment:

We neglected the KE_{spring}, which is small but could be adding intertia to the system and slowing down the oscillations. We can account for this:

<u>Improved mass-spring model</u>
Normalize $TE = KE + PE = 0$ for mass hanging in equilibrium position, at rest. Then for system in motion,

$$KE + PE = KE_{mass} + KE_{spring} + PE_{work}.$$

$$PE_{work} = \int_0^x k\, s \, \mathrm{d}s = \frac{1}{2} k\, x^2, \quad KE_{mass} = \frac{1}{2} m\, (x'(t))^2, \quad KE_{spring} = ????$$

How to model KE_{spring} ? Spring is at rest at top (where it's attached to bar), moving with velocity $x'(t)$ at bottom (where it's attached to mass). Assume it's moving with velocity $\mu\, x'(t)$ at location which is fraction μ of the way from the top to the mass. Then we can compute KE_{spring} as an integral with respect to μ, as the fraction varies $0 \le \mu \le 1$:

$$KE_{spring} = \int_0^1 \frac{1}{2} \left(\mu\, x'(t)\right)^2 \left(m_{spring}\, \mathrm{d}\mu\right)$$

$$= \frac{1}{2} m_{spring} (x'(t))^2 \int_0^1 \mu^2 d\mu = \frac{1}{6} m_{spring} (x'(t))^2 .$$

Thus

$$TE = \frac{1}{2}\left(m + \frac{1}{3} m_{spring}\right)(x'(t))^2 + \frac{1}{2} k\, x^2 = \frac{1}{2} M (x'(t))^2 + \frac{1}{2} k\, x^2 ,$$

where

$$M = m + \frac{1}{3} m_{spring}$$

$D_t(TE) = 0 \Rightarrow$

$$M\, x'(t)\, x''(t) + k\, x(t)\, x'(t) = 0 .$$
$$x'(t)\,(M\, x'' + k\, x) = 0 .$$

Since $x'(t) = 0$ only at isolated t-values, we deduce that the corrected equation of motion is

$$(M\, x'' + k\, x) = 0$$

with

$$\omega_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m + \frac{1}{3} m_{spring}}} .$$

Does this lead to a better comparison between model and experiment?

> $ms := .011;$ *# spring has mass 11g*
$M := m + \frac{1}{3} \cdot ms;$ *# "effective mass"*

$$ms := 0.011$$
$$M := 0.1036666667 \quad \textbf{(5)}$$

> $\omega := \sqrt{\frac{k}{M}};$ *# predicted angular frequency*
$f := evalf\left(\frac{\omega}{2 \cdot \mathrm{Pi}}\right);$ *# predicted frequency*
$T := \frac{1}{f};$ *# predicted period*

$$\omega := 5.559883146$$
$$f := 0.8848828855$$
$$T := 1.130093051 \quad \textbf{(6)}$$

>

Math 2250-004
Wed Mar 22

covering this on Tuesday :)

Wed: (3) finish long parts of today's notes
(4) introduce 5.6: forced oscillations
(5) quiz

Wednesday:
(1) • Please pick up your graded work packets!!
(2) • My office hours today are canceled.

Section 5.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems).

y_P: any single particular sol'n.
y_H: general homogeneous soltns.

There are two methods we will use:

• The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 5.5.

• The method of variation of parameters is more general, and yields an integral formula for a particular solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f. It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$. We'll study the case $n = 2$ only.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in the notes.

Exercise 1) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3 .$$

Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10, b_2 = 3$.

$$L(d_1 x + d_2) = b_1 x + b_2 \overset{\text{want}}{=} 10x + 3$$

$$\begin{aligned}
-5\,(\ y_P &= d_1 x + d_2\ )\\
+4\,(\ y_P' &= d_1\ )\\
+1\,(\ y_P'' &= 0\ )\\
\hline
L(y_P) &= x(-5d_1) + 1(-5d_2 + 4d_1) \overset{\text{want}}{=} 10x + 3\cdot 1
\end{aligned}$$

match coeffs of x: $-5d_1 = 10$
" " " 1: $-5d_2 + 4d_1 = 3$

$\Rightarrow d_1 = -2$; $-5d_2 - 8 = 3$
$-5d_2 = 11$
$d_2 = -\frac{11}{5}$

$$\boxed{y_P(x) = -2x - \frac{11}{5}}$$

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4y' - 5y = 10x + 3$$

$y = y_P + y_H$

$\uparrow$

$y'' + 4y' - 5y = 0$

$p(r) = r^2 + 4r - 5$

$= (r+5)(r-1)$

roots $r = -5, 1$.

$y = y_P + y_H$

$$\boxed{y = -2x - \tfrac{11}{5} + c_1 e^{-5x} + c_2 e^{x}}$$

$L(y) = f$

$\left.\begin{array}{l} L(y_P) = f \\ L(y_H) = 0 \end{array}\right\} \Rightarrow L(y_P + y_H)$

$= L(y_P) + L(y_H)$

$= f + 0$

$= f$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4y' - 5y = \boxed{14e^{2x}}$$

Hint: try $y_P = d\,e^{2x}$ because L transforms functions of that form into ones of the form $b\,e^{2x}$, i.e. $L\left(d\,e^{2x}\right) = b\,e^{2x}$. "$d$" is your "undetermined coefficient" for $b = 14$.

$-5\,[\,y_P = de^{2x}\,]$

$+4\,[\,y_P' = 2de^{2x}\,]$

$1\,[\,y_P'' = 4de^{2x}\,]$

$L(y_P) = de^{2x}(-5 + 8 + 4)$

$= 7de^{2x}$

$\overset{\text{want}}{=} 14e^{2x}$

$7d = 14$

$d = 2$

$$\boxed{y_P(x) = 2e^{2x}}$$

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4y' - 5y = 14e^{2x} - 20x - 6 .$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4y' - 5y = 14e^{2x} - 20x - 6$$
$$y(0) = 4$$
$$y'(0) = -4 .$$

4c) Check your answer with technology.

$L(2e^{2x}) = 14e^{2x}$

$L(-2x - \frac{11}{5}) = 10x + 3$

4a) $L(2e^{2x} - 2(-2x - \frac{11}{5}))$

$= L(2e^{2x}) - 2L(-2x - \frac{11}{5})$

$= 14e^{2x} - 2(10x + 3)$

$= 14e^{2x} - 20x - 6.$

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> with(DEtools):
> dsolve({y''(x) + 4·y'(x) − 5·y(x) = 14·e^(2·x) − 20·x − 6, y(0) = 4, y'(0) = −4});
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$$y(x) = \underbrace{\frac{8}{5}}_{c_2} e^{-5x} \underbrace{-4}_{c_1} e^{x} + \underbrace{2e^{2x} + 4x + \frac{22}{5}}_{y_P} \qquad (7)$$

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4a) cont'd.

$y = y_P + y_H$

$$\boxed{y = 2e^{2x} + 4x + \frac{22}{5} + c_1 e^{x} + c_2 e^{-5x}}$$

$y(0) = 4 = 2 + \frac{22}{5} + c_1 + c_2$

$y'(0) = -4 = 4 + 4 + c_1 - 5c_2$

Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = \boxed{2\cos(3x)}.$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \to V$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d\, e^{2x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why $V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?

$V = span\{\cos 3x, \sin 3x\}$ $\quad L: V \to V$

$y_P = d_1 \cos 3x + d_2 \sin 3x$

pick up here Wed

$$\begin{aligned}
-5\,[\, y_P &= d_1\cos 3x + d_2 \sin 3x \,]\\
+4\,[\, y' &= -3d_1 \sin 3x + 3d_2\cos 3x \,]\\
1\,[\, y'' &= -9d_1\cos 3x - 9d_2 \sin 3x \,]
\end{aligned}$$

$$L(y) = \cos 3x\,(-5d_1 + 12d_2 - 9d_1) + \sin 3x\,(-5d_2 - 12d_1 - 9d_2)$$

$$\underset{=}{\text{want}}\ (\cos 3x)(2) + (\sin 3x)(0)$$

sys for d_1, d_2:

$$\begin{aligned}
-14d_1 + 12d_2 &= 2\\
-12d_1 - 14d_2 &= 0
\end{aligned}$$

$$\begin{aligned}
-7d_1 + 6d_2 &= 1\\
6d_1 + 7d_2 &= 0
\end{aligned}$$

$$\begin{bmatrix} -7 & 6 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{-85}\begin{bmatrix} 7 & -6 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{7}{85} \\ +\frac{6}{85} \end{bmatrix}$$

wow!

$$\boxed{y_P(x) = -\frac{7}{85}\cos 3x + \frac{6}{85}\sin 3x}$$

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> with(DEtools) :
  dsolve(y''(x) + 4·y'(x) − 5·y(x) = 2·cos(3·x), y(x));
```

$$y(x) = \underbrace{e^{-5x}\_C2 + e^{x}\_C1}_{y_H} \underbrace{- \frac{7}{85}\cos(3x) + \frac{6}{85}\sin(3x)}_{\text{the } y_P \text{ we found.}} \qquad \textbf{(8)}$$

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All of the previous exercises rely on:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that

(i) $L : V \to V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

why: We already know this fact is true for matrix transformations $L(\underline{x}) = A_{n \times n}\underline{x}$ with $L : \mathbb{R}^n \to \mathbb{R}^n$ (because if the only homogeneous solution is $\underline{x} = \underline{0}$ then A reduces to the identity, so also each matrix equation $A\,\underline{x} = \underline{b}$ has a unique solution $\underline{x}$. The theorem above is a generalization of that fact to general linear transformations. There is an "appendix" explaining this at the end of today's notes, for students who'd like to understand the details.

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

(assuming the only such solution in your specified subspace that would solve the homogeneous DE is the zero solution):

6a) $L(y) = x^3 + 6x - 5$

$L : V \to V$

$f \in V$

$y_P = d_1 x^3 + d_2 x^2 + d_3 x + d_4 \cdot 1$

i.e. $V = \text{span}\{x^3, x^2, x, 1\}$

6b) $L(y) = 4e^{2x}\sin(3x)$

$V = \text{span}\{e^{2x}\sin 3x,\ e^{2x}\cos 3x\}$

$y_P = d_1 e^{2x}\sin 3x + d_2 e^{2x}\cos 3x$

$y_P' = b_1 e^{2x}\sin 3x + b_2 e^{2x}\cos 3x$

etc!

6c) $L(y) = x\cos(2x)$

$V = \text{span}\{x\cos 2x,\ \cos 2x,\ x\sin 2x,\ \sin 2x\}$

$y_P = d_1 x\cos 2x + d_2 \cos 2x + d_3 x\sin 2x + d_4 \sin 2x$

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> dsolve(y''(x) + 4·y'(x) - 5·y(x) = x·cos(2·x), y(x));
```

$$y(x) = \underbrace{e^{-5x}\_C2 + e^{x}\_C1}_{y_H} + \frac{1}{21025}(-1305x + 508)\cos(2x) + \frac{1}{21025}(1160x + 644)\sin(2x) \qquad (9)$$

e.g. $d_3 = \dfrac{1160}{21025}$

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BUT LOOK OUT

Exercise 7a) Find a particular solution to

$$y'' + 4y' - 5y = 4e^x .$$

$p(r) = r^2 + 4r - 5$
$= (r+5)(r-1)$
roots $r = -5, +1$

Hint: since $\boxed{y_H = c_1 e^x + c_2 e^{-5x}}$, a guess of $y_P = a e^x$ will not work (and $span\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\, x\, e^x$$

and factor $L = D^2 + 4D - 5I = [D + 5I] \circ [D - I]$.

$L(y) = [D^2 + 4D - 5I]y$
$= y'' + 4y' - 5y$

let's try $y_p = de^x$
$L(y_p) = 0$

$[D - aI]\, g(x)e^{ax} = g'(x)e^{ax}$
$g' e^{ax} + \cancel{g a e^{ax}} - \cancel{a g e^{ax}}$

$y_p = dxe^x$
$L(y_p) = [D+5I]\circ \underbrace{[D-I]\, dxe^x}$
$= [D+5I]\, de^x \quad (a=1)$
$= de^x + 5de^x$
$= 6de^x \overset{\text{want}}{=} 4e^x$
$d = 2/3$

$y_p = \frac{2}{3} x e^x$

OR

$-5\,[y_p = dxe^x]$
$+4\,[y_p' = de^x + dxe^x]$
$y_p'' = de^x + de^x + dxe^x$
$1\,[y_p'' = 2de^x + dxe^x]$

$L(y_p) = xe^x(\cancel{-5d + 4d + d}) \to 0$
$e^x(0 + 4d + 2d)$
$= 6de^x = 4e^x$
$6d = 4$
$d = \frac{2}{3}$
$y_p = \frac{2}{3} xe^x$ ✓

7b) check work with technology

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> with(DEtools):
> dsolve(y''(x) + 4·y'(x) - 5·y(x) = 4·e^x, y(x));
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$$y(x) = e^{-5x}\, \_C2 + e^x\, \_C1 + \frac{2}{3}\, x\, e^x \qquad \textbf{(10)}$$

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A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in lab you're working on this week) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in pages 341-346 of the text, and the table on page 346, reproduced here.

<u>Method of undetermined coefficients (extended case)</u>: If L has a factor $(\mathrm{D} - r\,I)^s$ and $e^{r\,x}$ is also associated with (a portion of) the right hand side $f(x)$ then the corresponding guesses you would have made in the "base case" need to be multiplied by x^s , as in <u>Exercise 7</u>. (If you understood the homework problem last week about factoring L into composition of terms like $\boxed{(\mathrm{D} - r\,I)^s}$, then you have an inkling of why this recipe works. If you didn't understand that last week problem, there's another one this week so you get a second chance. :-)) You may also need to use superposition, as in <u>Exercise 4</u>, if different portions of $f(x)$ are associated with different exponential functions.

Extended case of undetermined coefficients

$f(x)$	multiply by y_P	$s > 0$ when $p(r)$ has these roots:
$P_m(x) = b_0 + b_1 + \ldots + b_m x^m$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right)$	$r = 0$
$b_1 \cos(\omega\,x) + b_2 \sin(\omega\,x)$	$x^s\left(c_1 \cos(\omega\,x) + c_2 \sin(\omega\,x)\right)$	$r = \pm\, i\,\omega$
$e^{a\,x}\left(b_1 \cos(\omega\,x) + b_2 \sin(\omega\,x)\right)$	$x^s e^{a\,x}\left(c_1 \cos(\omega\,x) + c_2 \sin(\omega\,x)\right)$	$r = a \pm i\omega$
$b_0 e^{a\,x}$	$x^s c_0 e^{a\,x}$	$r = a$
$\left(b_0 + b_1 + \ldots + b_m x^m\right) e^{a\,x}$	$x^s\left(c_0 + c_1 x + c_2 x^2 + \ldots + c_m x^m\right) e^{a\,x}$	$r = a$

e.g. (exercise 7) $y'' + 4y' - 5y = 4e^x$

• multi by x^s so that no terms in guess solve homog DE

Exercise 8) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

8a) $y''' + 2y'' = x^2 + 6x$
(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

> dsolve$\left(y'''(x) + 2\cdot y''(x) = x^2 + 6\cdot x, y(x)\right)$;

$$y(x) = \frac{1}{24}x^4 + \frac{5}{12}x^3 + \frac{1}{4}e^{-2x}\_C1 - \frac{5}{8}x^2 + \_C2\,x + \_C3 \qquad \textbf{(11)}$$

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8b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$
(So the characteristic polynomial for $L(y) = 0$ is
$r^2 - 4r + 13) = (r-2)^2 + 9 = (r-2+3i)(r-2-3i)$.)

> dsolve$\left(y''(x) - 4\cdot y'(x) + 13\cdot y(x) = 4\cdot e^{2\cdot x}\cdot\sin(3\cdot x), y(x)\right)$;

$$y(x) = e^{2x}\sin(3x)\,\_C2 + e^{2x}\cos(3x)\,\_C1 - \frac{2}{3}e^{2x}\cos(3x)\,x \qquad \textbf{(12)}$$

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8c) $y'' + 5y' + 4y = 5\cos(2x) + 4e^x + 5e^{-x}$.
(So the characteristic polynomial for $L(y) = 0$ is $p(r) = r^2 + 5r + 4 = (r+4)(r+1)$.)

> dsolve$\left(y''(x) + 5\cdot y'(x) + 4\cdot y(x) = 5\cdot\cos(2\cdot x) + 4\cdot e^x + 5\cdot e^{-x}, y(x)\right)$;

$$y(x) = e^{-x}\_C2 + e^{-4x}\_C1 + \frac{1}{2}\sin(2x) + \frac{2}{5}e^x + \frac{5}{3}e^{-x}x - \frac{5}{9}e^{-x} \qquad \textbf{(13)}$$

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Variation of Parameters: This is an alternate method for finding particular solutions. Its advantage is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), ... y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x) y^{(n-1)} + ... + p_1(x) y' + p_0(x) y = 0 .$$

Then $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) + \ ...\ + u_n(x) y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), ... u_n(x)$ have derivatives satisfying the matrix equation

$$\begin{bmatrix} u_1' \\ u_2' \\ : \\ u_n' \end{bmatrix} = \left[W\left(y_1, y_2, ..., y_n\right) \right]^{-1} \begin{bmatrix} 0 \\ 0 \\ : \\ f \end{bmatrix}$$

where $\left[W\left(y_1, y_2, ..., y_n\right) \right]$ is the Wronskian matrix.

Here's how to check this fact when $n = 2$: Write

$$y_p = y = u_1 y_1 + u_2 y_2 .$$

Thus

$$y' = u_1 y_1' + u_2 y_2' + \left(u_1' y_1 + u_2' y_2\right) .$$

Set

$$\left(u_1' y_1 + u_2' y_2\right) = 0.$$

Then

$$y'' = u_1 y_1'' + u_2 y_2'' + \left(u_1' y_1' + u_2' y_2'\right) .$$

Set

$$\left(u_1' y_1' + u_2' y_2'\right) = f.$$

Notice that the two $(\ ...\ )$ equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p. Under these conditions we compute

$$\begin{aligned} & p_0 \left[y = u_1 y_1 + u_2 y_2 \right] \\ & + p_1 \left[y' = u_1 y_1' + u_2 y_2' \right] \\ & + 1 \left[y'' = u_1 y_1'' + u_2 y_2'' + f \right] \\ & \quad L(y) = u_1 L\left(y_1\right) + u_2 L\left(y_2\right) + f \\ & L(y) = 0 + 0 + f = f \end{aligned}$$

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." We will not discuss these in class, but I'll be happy to chat about the arguments with anyone who's interested, outside of class. They only use ideas we've talked about already, although they are abstract.

Theorem 0:

- Let V and W be vector spaces. Let V have dimension $n < \infty$ and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$ be a basis for V.
- Let $L : V \to W$ be a linear transformation, i.e. $L(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = L(\underline{\boldsymbol{u}}) + L(\underline{\boldsymbol{v}})$ and $L(c\,\underline{\boldsymbol{u}}) = c\,L(\underline{\boldsymbol{u}})$ holds $\forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in V, c \in \mathbb{R}$.) Consider the range of L, i.e.

$$\begin{aligned} Range(L) &:= \{L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n) \in W, \text{ such that each } d_j \in \mathbb{R}\} \\ &= \{d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + \ldots + d_nL(\underline{\boldsymbol{v}}_n) \in W, \text{ such that each } d_j \in \mathbb{R}\} \\ &= span\{L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), \ldots L(\underline{\boldsymbol{v}}_n)\}. \end{aligned}$$

Then $Range(L)$ is $n-dimensional$ if and only if the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$.

proof:

(i) $\Leftarrow$: The only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$ implies $Range(L)$ is $n-dimensional$:

If we can show $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), \ldots L(\underline{\boldsymbol{v}}_n)$ are linearly independent, then they will be a basis for $Range(L)$ and this subspace will have dimension n. So, consider the dependency equation:

$$d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + \ldots + d_nL(\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}}.$$

Because L is a linear transformation, we can rewrite this equation as

$$L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}}.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n = \underline{\boldsymbol{0}}.$$

Since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ are a basis they are linearly independent, so $d_1 = d_2 = \ldots = d_n = 0$.

$\square$

(ii) $\Rightarrow$: $Range(L)$ is $n-dimensional$ implies the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$: Since the range of L is $n-dimensional$, $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), \ldots L(\underline{\boldsymbol{v}}_n)$ must be linearly independent. Now, let $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ be a homogeneous solution, $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$. In other words,

$$\begin{aligned} &L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}} \\ \Rightarrow\ &d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + \ldots + d_nL(\underline{\boldsymbol{v}}_n) = \underline{\boldsymbol{0}} \\ \Rightarrow\ &d_1 = d_2 = \ldots = d_n = 0 \Rightarrow \underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}. \end{aligned}$$

$\square$

Theorem 1 Let Let V and W be vector spaces, both with the same dimension $n < \infty$. Let $L : V \to W$ be a linear transformation. Let the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}}$ be $\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$. Then for each $\underline{\boldsymbol{w}} \in W$ there is a unique $\underline{\boldsymbol{v}} \in V$ with $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$.

proof: By Theorem 0, the dimension of $Range(L)$ is $n-dimensional$. Therefore it must be all of W. So for each $\underline{\boldsymbol{w}} \in W$ there is at least one $\boldsymbol{v}_P \in V$ with $L(\underline{\boldsymbol{v}}_P) = \underline{\boldsymbol{w}}$. But the general solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{v}}_P + \underline{\boldsymbol{v}}_H$, where $\underline{\boldsymbol{v}}_H$ is the general solution to the homogeneous equation. By assumption, $\underline{\boldsymbol{v}}_H = \underline{\boldsymbol{0}}$, so the particular solution is unique.

□

Remark: In the base case of undetermined coefficients, $W = V$. In the extended case, W is the space in which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s. This is because if L factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m}$$

and if f is in a subspace W associated with the characteristic polynomial root r_m, then for $s = k_m$ the factor $(D - r_m I)^{k_m}$ of L will transform the space $V = x^s W$ back into W, and not transform any non-zero function in V into the zero function. And the other factors of L will then preserve W, also without transforming any non-zero elements to the zero function.

Friday
- start details of today's notes

Monday
: Finish Fri. notes
- talk about pendulum model

Tuesday
- the experiments
- start Laplace transforms

Wed
- Laplace
- review notes

Math 2250-004
Friday Mar 24

tomorrow Thurs in labs:
$y(x) \longrightarrow x(t)$

Section 5.6: forced oscillations in mechanical (and electrical) systems. We will continue to discuss section 5.6 on Monday using these notes.
Overview for solutions $x(t)$ to

$$\boxed{m\, x'' + c\, x' + k\, x = F_0 \cos(\omega\, t)}$$

using section 5.5 undetermined coefficients algorithms.

- $\boxed{\text{undamped } (c = 0):}$
In this case the complementary homogeneous differential equation for $x(t)$ is

$$m\, x'' + k\, x = 0$$
$$x'' + \frac{k}{m} x = 0$$
$$x'' + \omega_0^2\, x = 0$$

$$\boxed{\begin{aligned} m x'' + k x &= F_0 \cos \omega t \\ x'' + \frac{k}{m} x &= \frac{F_0}{m} \cos \omega t \\ x'' + \omega_0^2 x &= \frac{F}{m} \cos \omega t \end{aligned}}$$

which has simple harmonic motion solutions $x_H(t) = C \cos\left(\omega_0 t - \alpha\right)$. So for the non homongeneous DE the method of undetermined coefficients implies we can find particular and general solutions as follows:

$x_p = A \cos \omega t + \cancel{B \sin \omega t}$

- $\omega \neq \omega_0 := \sqrt{\frac{k}{m}} \quad \Rightarrow x_P = A \cos(\omega\, t)$ because only even derivatives, we don't need $\sin(\omega\, t)$ terms !!

$$\boxed{\Rightarrow x = x_P + x_H = A \cos(\omega\, t) + C_0 \cos\left(\omega_0 t - \alpha_0\right)}$$

- $\omega \neq \omega_0$ but $\omega \approx \omega_0$, $C \approx C_0$ Beating!
- $\omega = \omega_0 \qquad \Rightarrow \boxed{x_P = t\left(A \cos\left(\omega_0\, t\right) + B \sin\left(\omega_0\, t\right)\right)}$

$$\Rightarrow x = x_P + x_H = C\, t \cos\left(\omega_0\, t - \alpha\right) + C_0 \cos\left(\omega_0 t - \alpha_0\right).$$

("pure" resonance!)

$x'' + \omega_0^2 x = \frac{F}{m} \cos \omega_0 t$

pure reson.

....

$$\boxed{m\, x'' + c\, x' + k\, x = F_0 \cos(\omega\, t)}$$

- $\boxed{\text{damped } (c > 0):}$ in all cases $x_P = A \cos(\omega\, t) + B \sin(\omega\, t) = C \cos(\omega\, t - \alpha)$ (because the roots of the characteristic polynomial are never $\pm i\, \omega$ when $c > 0$).
  - underdamped: $\quad x = x_P + x_H = C \cos(\omega\, t - \alpha) + e^{-p\, t} C_1 \cos\left(\omega_1 t - \alpha_1\right)$.
  - critically-damped: $x = x_P + x_H = C \cos(\omega\, t - \alpha) + e^{-p\, t}\left(c_1 t + c_2\right)$.
  - over-damped: $\quad x = x_P + x_H = C \cos(\omega\, t - \alpha) + c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

$\uparrow$
steady periodic part of the soln

$x_{sp}(t) = x_p(t)$

$x_H(t) \to 0$ exp.

$x_H(t) = x_{tr}(t)$
transient part.

take-home quiz
hand in at labs tomorrow

- in all three <u>damped</u> cases on the previous page, $x_H(t) \to 0$ exponentially and is called the <u>transient solution</u> $x_{tr}(t)$ (because it disappears as $t \to \infty$. And in <u>these damped cases</u> $x_P(t)$ as above is called the <u>steady periodic solution</u> $x_{sp}(t)$ (because it is what persists as $t \to \infty$, and because it's periodic).

- if c is small enough and $\omega \approx \omega_0$ then the amplitude C of $x_{sp}(t)$ can be large relative to $\frac{F_0}{m}$, and the system can exhibit <u>practical resonance.</u> This can be an important phenomenon in electrical circuits, where amplifying signals is important.

<u>forced undamped oscillations</u>:

<u>Exercise 1a)</u> Solve the initial value problem for $x(t)$:

$$x'' + 9x = 80\cos(5t)$$
$$x(0) = 0$$
$$x'(0) = 0 .$$

1b) This superposition of two sinusoidal functions is periodic because there is a common multiple of their (shortest) periods. <u>What is this</u> (common) period?

1c) Compare your solution and reasoning with the display at the bottom of this page.

$m x'' + k x = F_0 \cos \omega t$

$x'' + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t$

ω_0^2

$x_H: \quad x'' + 9x = 0$

$x_H(t) = A\cos 3t + B \sin 3t$

$\left(p(r) = r^2 + 9 = 0 \quad r^2 = -9 \quad r = \pm 3i \right)$

$+9\,[\,x_p = C\cos(5t)\,]$
$+0\,[\,x' = -5C\sin(5t)\,]$
$1\,[\,x'' = -25C\cos 5t\,]$

$x'' + 9x = \cos 5t\,\underbrace{(9C - 25C)}_{-16C} \overset{\text{want}}{=} 80\cos 5t$

<u>need</u> $-16C = 80$
$C = -5$

$\boxed{x_p = -5\cos 5t}$

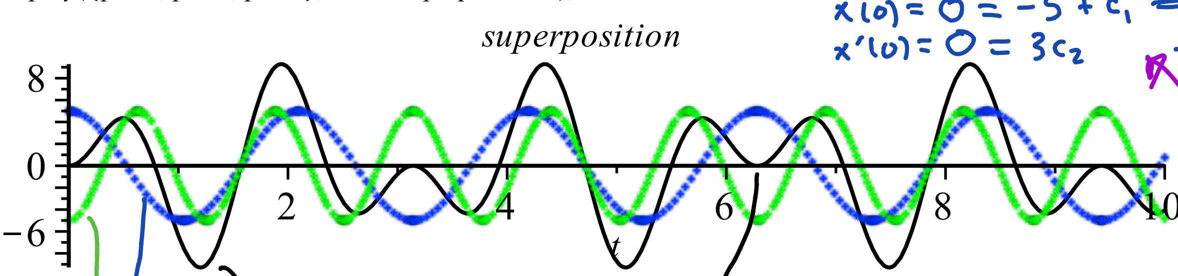
$\boxed{x(t) = -5\cos 5t + 5\cos 3t}$

$x(0) = 0$
$x'(0) = 0$ ✓

$x(t) = x_p + x_H$
$x(t) = -5\cos 5t + c_1\cos 3t + c_2 \sin 3t$
$x(0) = 0 = -5 + c_1 \Rightarrow c_1 = 5$
$x'(0) = 0 = 3c_2 \Rightarrow c_2 = 0$

$(x'(t) = +25\sin 5t - 3c_1 \sin 3t + 3c_2\cos 3t$
$x(0) = 0 + 0 + 3c_2)$

```
> with(plots):
> plot1 := plot(-5·cos(5·t), t = 0..10, color = green, style = point):
  plot2 := plot(5·cos(3·t), t = 0..10, color = blue, style = point):
  plot3 := plot(-5·cos(5·t) + 5·cos(3·t), t = 0..10, color = black):
  display({plot1, plot2, plot3}, title='superposition');
```



period of soltn $x(t)$?

$x(t) = -5\cos 5t + 5\cos 3t$

$T_1 = \frac{2\pi}{5} \qquad T_2 = \frac{2\pi}{3}$

least common multiple $\left(\frac{2\pi}{5}, \frac{2\pi}{3}\right) = 2\pi$

$\cdot 15:\ n_1\left(\cancel{\frac{2\pi}{5}}\right) = n_2\left(\frac{2\pi}{3}\right) \quad n_1, n_2 \in \mathbb{N}$

$3n_1 = 5n_2$

$n_1 = 5$
$n_2 = 3$

$5\left(\frac{2\pi}{5}\right)$ or $3\left(\frac{2\pi}{3}\right)$

2π

In general:

undamped forced IVP, $\omega \neq \omega_0$, with letters

$$\begin{cases} x'' + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

$+\frac{k}{m}\,(x_p = A\cos\omega t)$
$+\,0\,(x_p' = -A\omega\sin\omega t)$
$+\,1\,(x_p'' = -A\omega^2\cos\omega t)$

$$L(x_p) = \cos\omega t\, A\Big[\underbrace{\tfrac{k}{m}}_{\omega_0^2} - \omega^2\Big] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega t$$

deduce $A(\omega_0^2 - \omega^2) = \frac{F_0}{m}$. $\quad A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$

$A = \frac{F_0}{m}/(\omega_0^2 - \omega^2)$

So,
$$\boxed{x_p(t) = -\frac{F_0}{m}\frac{1}{(\omega^2 - \omega_0^2)}\cos\omega t}$$
Note $x_H(t) = A\cos\omega_0 t + B\sin\omega_0 t$.

so, by plugging in IC observation
IVP solution is

$$x(t) = \frac{+F_0}{m(\omega^2 - \omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t$$

$x_H \quad x_p \quad x_H \quad x_H$

check. NR!

$x = x_p + x_H$

$x = -\frac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t$

$x(0) = x_0 = -\frac{F_0}{m(\omega^2 - \omega_0^2)} + c_1$

$\boxed{c_1 = x_0 + \frac{F_0}{m(\omega^2 - \omega_0^2)}}$

$x'(0) = v_0 = \omega_0 c_2$

$\boxed{c_2 = \frac{v_0}{\omega_0}}$

There is an interesting beating phenomenon for $\omega \approx \omega_0$ (but still with $\omega \neq \omega_0$). This is explained analytically via trig identities, and is familiar to musicians in the context of superposed sound waves (which satisfy the homogeneous linear "wave equation" partial differential equation):

$$\begin{aligned} \cos(\underbrace{\alpha - \beta}_{\omega_0 t}) - \cos(\underbrace{\alpha + \beta}_{\omega t}) &= \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \\ &\quad - (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) \\ &= 2\sin(\alpha)\sin(\beta)\ . \end{aligned}$$

Set $\alpha = \frac{1}{2}(\omega + \omega_0)t$, $\beta = \frac{1}{2}(\omega - \omega_0)t$ in the identity above, to rewrite the first term in $x(t)$ as a product rather than a difference:

$T_1 = \frac{2\pi}{\bar{\omega}}$ $\qquad T_2 = \frac{2\pi}{\frac{1}{2}|\omega - \omega_0|} = \frac{4\pi}{|\omega - \omega_0|}$

$$x(t) = \frac{F_0}{m(\omega^2 - \omega_0^2)}2\sin\left(\frac{1}{2}(\omega + \omega_0)t\right)\sin\left(\frac{1}{2}(\omega - \omega_0)t\right) + x_0\cos(\omega_0 t) + \frac{v_0}{\omega_0}\sin(\omega_0 t)\ .$$

In this product of sinusoidal functions, the first one has angular frequency and period close to the original angular frequencies and periods of the original sum. But the second sinusoidal function has small angular frequency and long period, given by

$$\text{angular frequency: } \frac{1}{2}(\omega - \omega_0), \qquad \text{period: } \frac{4\pi}{|\omega - \omega_0|}\ .$$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots$

as $\omega \to \omega_0$ get resonance formula

$$x(t) \longrightarrow \frac{F_0\ \cancel{2}}{m\cancel{(\omega - \omega_0)}(\omega + \omega_0)}(\sin\omega_0 t)\left[\cancel{\tfrac{1}{2}}\cancel{(\omega - \omega_0)}t \pm \frac{\theta^3}{3!}\right]$$

$$\longrightarrow \boxed{\frac{F_0}{2m\omega_0}t\sin\omega_0 t}$$

$$x(t) = \frac{F_0}{m(\omega^2-\omega_0^2)}\left(\cos\omega_0 t - \cos\omega t\right) = \frac{F_0}{m(\omega^2-\omega_0^2)}\, 2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

We will call half that period the beating period, as explained by the next exercise: (when $x_0 = 0,\ v_0 = 0$)

$$\text{beating period: } \frac{2\pi}{|\omega-\omega_0|}, \qquad \text{beating amplitude: } \frac{2F_0}{m\left|\omega^2-\omega_0^2\right|}.$$

Exercise 2a) Use one of the formulas on the previous page to write down the IVP solution $x(t)$ to

$$x'' + 9x = 80\cos(3.1\, t) \qquad x'' + \omega_0^2 x = \frac{F_0}{m}\cos(3.1t)$$

$$\left.\begin{aligned} x(0) &= 0 \\ x'(0) &= 0 . \end{aligned}\right\}$$

2b) Compute the beating period and amplitude. Compare to the graph shown below.

$\frac{F_0}{m} = 80$

$\omega_0 = 3$

$\omega = 3.1$

$(3.1)^2 - 3^2 = (3.1-3)(3.1+3)$

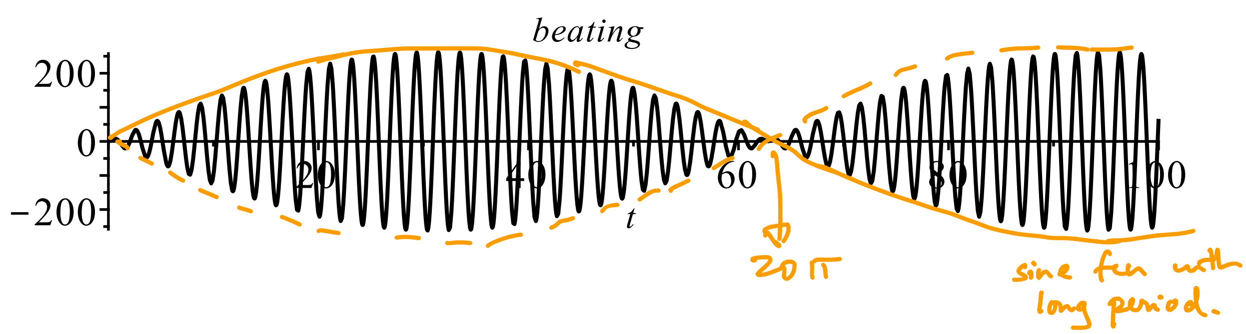
$$x(t) = 80\,\frac{1}{.61}\left(\cos 3t - \cos 3.1t\right)$$

$$= \left(\frac{80}{.61}\cdot 2\right)\sin(3.05t)\,\sin(\underbrace{.05t}_{})$$

$$T_2 = \frac{2\pi}{.05} = 40\pi \approx 120$$

beating period $= 20\pi$!

```
> plot(262.3·sin(3.05·t)sin(.05·t), t=0..100, color=black, title=`beating`);
```



Resonance:

Resonance! $\omega = \omega_0$ (and the limit as $\omega \to \omega_0$)

$$\begin{cases} x'' + \omega_0^2 x = \frac{F_0}{m}\cos\omega_0 t \\ x(0) = x_0 \\ x'(0) = v_0 \end{cases}$$

using §5.5, guess

$$+\omega_0^2 (\quad x_p = t\,(A\cos\omega_0 t + B\sin\omega_0 t) \quad)$$

$$0 (\quad x_p' = t\,(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + A\cos\omega_0 t + B\sin\omega_0 t \quad)$$

$$+1 (\quad x_p'' = t\,(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + \left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right]2 \quad)$$

$$L(x_p) = t\,(0) + 2\left[-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right] \overset{\text{want}}{=} \frac{F_0}{m}\cos\omega_0 t$$

Deduce $A = 0$

$B = \frac{F_0}{2m\omega_0}$

$$\underbrace{\boxed{x_p(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t}}_{\text{sats } x(0)=0,\ x'(0)=0}$$

, so IVP soln is

$$\boxed{x(t) = \frac{F_0}{2m\omega_0}\, t\sin\omega_0 t + x_0\cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t}$$

You can also get this solution by letting $\omega \to \omega_0$ in the beating formula. We will probably do it that way in class.

$$x(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t + x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

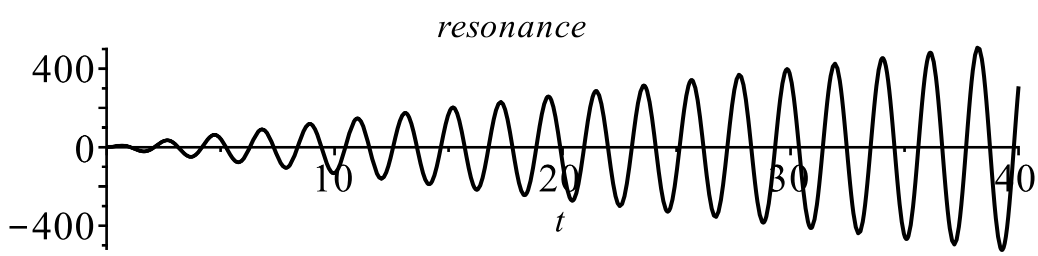
<u>Exercise 3a)</u> Solve the IVP

$$x'' + 9x = 80\cos(3t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

First just use the general solution formula above this exercise and substitute in the appropriate values for the various terms. Then, if time, use variation of parameters (see the last pages of today's notes), to check a particular solution and to illustrate this alternate method for finding particular solutions.

<u>3b)</u> Compare the solution graph below with the beating graph in <u>exercise 2</u>.

> $plot\left(\frac{40}{3} t \cdot \sin(3 \cdot\ t),\ t = 0..40,\ color = black,\ title = \text{`resonance`}\right);$



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