Wed 3/8: Use Tuesday notes
& introduce Wed notes
• quiz at end of class

Friday: start here

Math 2250-004
Wed Mar 8

5.3 continued. How to find the solution space for n^{th} order linear homogeneous DE's with constant coefficients, and why the algorithms work.

$$L(y) = y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \dots + a_1 y' + a_0 y = 0$$

Strategy: In all cases we first try to find a basis for the n-dimensional solution space made of or related to exponential functions....trying $y(x) = e^{r x}$ yields

$$L(y) = e^{r x}\left(r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0\right) = e^{r x} p(r) . = 0$$

The characteristic polynomial $p(r)$ and how it factors are the keys to finding the solution space to $L(y) = 0$. There are three cases, of which the first two (distinct and repeated real roots) are in yesterday's notes.

Case 3) $p(r)$ has complex number roots. This is the hardest, but also most interesting case.The punch line is that exponential functions $e^{r x}$ still work, except that $r = a \pm b\, i$; but, rather than use those complex exponential functions to construct solution space bases we decompose them into real-valued solutions that are products of exponential and trigonometric functions.

To understand how this all comes about, we need to learn Euler's formula . This also lets us review some important Taylor's series facts from Calc 2. As it turns out, complex number arithmetic and complex exponential functions actually are important in many engineering and science applications.

Recall the Taylor-Maclaurin formula from Calculus

$$f(x) \sim \boxed{f(0)} + \boxed{f'(0)} x + \frac{1}{2!} f''(0) x^2 + \frac{1}{3!} f'''(0) x^3 + \dots + \frac{1}{n!} f^{(n)}(0) x^n + \dots$$

(Recall that the partial sum polynomial through order n matches f and its first n derivatives at $x_0 = 0$. When you studied Taylor series in Calculus you sometimes expanded about points other than $x_0 = 0$. You also needed error estimates to figure out on which intervals the Taylor polynomials actually coverged back to f.)

$f(x) \approx p_n(x)$ so that $f(0) = p(0)$, $f'(0) = p'(0)$ ... $f^{(n)}(0) = p^{(n)}(0)$.

$p_n(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

$f(0) = p(0) = a_0$ $\quad p'(x) = a_1 + 2a_2 x + 3a x^2 + \dots$

$f'(0) = p'(0) = a_1$

$f''(0) = p''(0) = 2a_2 \Rightarrow a_2 = \frac{1}{2} f''(0)$

$f'''(0) = p'''(0) = 6a \Rightarrow a_3 = \frac{1}{3!} f'''(0)$

Exercise 1) Use the formula above to recall the three very important Taylor series for

1a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$

1b) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

1c) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

In Calculus you checked that these series actually converge and equal the given functions, for all real numbers x.

$f(x) = e^x$
$f^{(n)}(x) = e^x$ | @ $x=0$: 1

$g(x) = \cos x$ | @ $x=0$: 1
$g' = -\sin x$ | 0
$g'' = -\cos x$ | -1
$g''' = \sin x$ | 0
$g^{(4)} = \cos x$

$h(x) = \sin x$ | @ $x=0$: 0
$h' = \cos x$ | 1
$h'' = -\sin x$ | 0
$h''' = -\cos x$ | -1
$h^{(4)} = \sin x$

Exercise 2) Let $x = i\,\theta$ and use the Taylor series for e^x as the definition of $e^{i\,\theta}$ in order to derive Euler's formula:

$$\boxed{e^{i\,\theta} = \cos(\theta) + i \sin(\theta)}\,.$$

$e^{i(\alpha+\beta)} = e^{i\alpha} e^{i\beta}$
HW

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \dots$$

$\cos\theta$ $\qquad i\sin\theta$

From Euler's formula it makes sense to define

$$e^{a+b\,i} := e^a e^{b\,i} = e^a(\cos(b) + i\sin(b))$$

for $a, b \in \mathbb{R}$. So for $x \in \mathbb{R}$ we also get

$$e^{rx} = e^{(a+b\,i)x} = e^{a\,x}(\cos(b\,x) + i\sin(b\,x)) = e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\,.$$

$e^{ax}e^{ibx}$ Euler

For a complex function $f(x) + i\,g(x)$ we define the derivative by

$$D_x(f(x) + i\,g(x)) := f'(x) + i\,g'(x)\,.$$

It's straightforward to verify (but would take some time to check all of them) that the usual differentiation rules, i.e. sum rule, product rule, quotient rule, constant multiple rule, all hold for derivatives of complex functions. The following rule pertains most specifically to our discussion and we should check it:

<u>Exercise 3)</u> Check that $D_x\left(e^{(a+b\,i)x}\right) = (a+b\,i)e^{(a+b\,i)x}$, i.e.

$$\boxed{D_x\, e^{r\,x} = r\,e^{r\,x}}$$

even if r is complex. (So also $D_x^2\, e^{r\,x} = D_x r\,e^{r\,x} = r^2\,e^{r\,x}$, $D_x^3\, e^{r\,x} = r^3\,e^{r\,x}$, etc.)

$$D_x\left(e^{(a+bi)x}\right) = D_x\left(e^{ax}e^{ibx}\right) = D_x\left(e^{ax}(\cos bx + i\sin bx)\right)$$
$$= D_x\left[e^{ax}\cos bx + i\,e^{ax}\sin bx\right]$$
$$D_x e^{rx} = e^{ax}(a\cos bx - b\sin bx) + i\,e^{ax}(a\sin bx + b\cos bx)$$
$$re^{rx} = (a+bi)(e^{ax}\cos bx + i\,e^{ax}\sin bx)$$

Now return to our differential equation questions, with

$$L(y) := y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1\,y' + a_0\,y\,.$$

Then even for complex $r = a + b\,i$ $(a, b \in \mathbb{R})$, our work above shows that

$$L\left(e^{r\,x}\right) = e^{r\,x}\left(r^n + a_{n-1}r^{n-1} + \ldots + a_1 r + a_0\right) = e^{r\,x}\,p(r)\,.$$

So if $r = a + b\,i$ is a complex root of $p(r)$ then $e^{r\,x}$ is a complex-valued function solution to $L(y) = 0$. But L is linear, and because of how we take derivatives of complex functions, we can compute in this case that

$$0 + 0\,i = L\left(e^{r\,x}\right) = L\left(e^{a\,x}\cos(b\,x) + i\,e^{a\,x}\sin(b\,x)\right)$$
$$= L\left(e^{a\,x}\cos(b\,x)\right) + i\,L\left(e^{a\,x}\sin(b\,x)\right).$$

$e^{ax}\cos bx + i\,e^{ax}\sin bx$ ||

Equating the real and imaginary parts in the first expression to those in the final expression (because that's what it means for complex numbers to be equal) we deduce

$$0 = L\left(e^{a\,x}\cos(b\,x)\right)$$
$$0 = L\left(e^{a\,x}\sin(b\,x)\right).$$

<u>Upshot:</u> If $r = a + b\,i$ is a complex root of the characteristic polynomial $p(r)$ then

$$y_1 = e^{a\,x}\cos(b\,x)$$
$$y_2 = e^{a\,x}\sin(b\,x)$$

are two solutions to $L(y) = 0$. (The conjugate root $a - b\,i$ would give rise to y_1, $-y_2$, which have the same span.

Case 3) Let L have characteristic polynomial

$$p(r) = r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0$$

with real constant coefficients $a_{n-1}, \ldots, a_1, a_0$. If $(r - (a + b\,i))^k$ is a factor of $p(r)$ then so is the conjugate factor $(r - (a - b\,i))^k$. Associated to these two factors are $2\,k$ real and independent solutions to $L(y) = 0$, namely

$$\begin{array}{c} e^{a x}\cos(b\,x),\ e^{a x}\sin(b\,x) \\ x\,e^{a x}\cos(b\,x),\ x\,e^{a x}\sin(b\,x) \\ \vdots \qquad\qquad \vdots \\ x^{k-1}e^{a x}\cos(b\,x),\ x^{k-1}e^{a x}\sin(b\,x) \end{array}$$

Combining cases 1,2,3, yields a complete algorithm for finding the general solution to $L(y) = 0$, as long as you are able to figure out the factorization of the characteristic polynomial $p(r)$.

Exercise 4) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 4\,y = 0\,.$$

(You were told a basis in the last problem of last week's hw....now you know where it came from.)

$$\boxed{\begin{array}{l} r = a \pm i b \\ y_1(x) = e^{ax}\cos bx \\ y_2(x) = e^{ax}\sin bx \end{array}}$$

$p(r) = r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$ $\quad a = 0$, $b = 2$

$\underline{\text{recipe}}$: $y_1(x) = \cos 2x$

$y_2(x) = \sin 2x$

Exercise 5) Find a basis for the solution space of functions $y(x)$ that solve

$$y'' + 6\,y' + 13\,y = 0\,.$$

$p(r) = r^2 + 6r + 13 = 0$

$(r+3)^2 + 4 = 0$

$(r+3)^2 = -4$

$r + 3 = \pm 2i$

$r = -3 \pm 2i$ $\quad a = -3$, $b = 2$

$y_1(x) = e^{-3x}\cos 2x$

$y_2(x) = e^{-3x}\sin 2x$

Exercise 6) Suppose a 7^{th} order linear homogeneous DE has characteristic polynomial

$$p(r) = (r^2 + 6\,r + 13)^2 (r - 2)^3\,.$$

What is the general solution to the corresponding homogeneous DE?

$\Big((\underbrace{(r+3)^2 + 4}_{-(2i)^2})^2\Big)(r-2)^3$

$= \big((r+3+2i)(r+3-2i)\big)^2 (r-2)^3$

$= (r+3+2i)^2 (r+3-2i)^2 (r-2)^3$

$\mathbb{C}$ roots $-3 \mp 2i$

$y_H(x) = c_1 e^{-3x}\cos 2x + c_2 e^{-3x}\sin 2x + c_3\,x e^{-3x}\cos 2x + c_4\,x e^{-3x}\sin 2x$
$+ c_5 e^{2x} + c_6\,x e^{2x} + c_7\,x^2 e^{2x}$

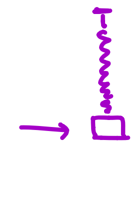
Friday: Wednesday notes
at end of class, introduce § 5.4

Friday Mar 10

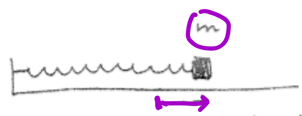
$\boxed{5.4:}$ Applications of 2^{nd} order linear homogeneous DE's with constant coefficients, to unforced spring (and related) configurations.

In this section we study the differential equation below for functions $x(t)$:

$$m\,x'' + c\,x' + k\,x = 0\,.$$



model



x(t) displacement from equilibrium.

equilibrium location

possible external forcing function

Newton's 2^{nd} Law: $\quad m\,x''(t) = \text{net forces}$

$$= \underbrace{-c\,x'}_{} \; \underbrace{-\,k\,x}_{} \; + \underbrace{F(t)}_{}$$

linear drag, proportional to velocity

Hooke's Law

force from compression or stretching of spring, relative to equilibrium $x = 0$

$$m\,x'' + c\,x' + k\,x = F(t)$$

§5.3: $y(x)$ $\quad$ §5.4: $x(t)$

§5.4: $F = 0$

In section 5.4 we assume the time dependent external forcing function $F(t) \equiv 0$. The expression for internal forces $-c\,x' - k\,x$ is a linearization model, about the constant solution $x = 0$, $x' = 0$, for which the net forces must be zero. Notice that $c \geq 0$, $k > 0$. The actual internal forces are probably not exactly linear, but this model is usually effective when $x(t), x'(t)$ are sufficiently small. k is called the Hooke's constant, and c is called the damping coefficient.

$mx'' + cx' + kx = 0$

This is a constant coefficient linear homogeneous DE, so we try $x(t) = e^{r\,t}$ and compute

$$L(x) := m\,x'' + c\,x' + k\,x = e^{r\,t}\left(m\,r^2 + c\,r + k\right) = e^{r\,t}\,p(r).$$

The different behaviors exhibited by solutions to this mass-spring configuration depend on what sorts of roots the characteristic polynomial $p(r)$ processes...

Case 1) no damping ($c = 0$).

$$m\,x'' + k\,x = 0$$

$$x'' + \frac{k}{m}\,x = 0\,.$$

$$p(r) = r^2 + \frac{k}{m}\,,$$

has roots

$$r^2 = -\frac{k}{m} \quad \text{i.e.} \quad r = \pm\,i\sqrt{\frac{k}{m}}\,.$$

So the general solution is

$$x(t) = c_1\cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2\sin\left(\sqrt{\frac{k}{m}}\,t\right).$$

We write $\sqrt{\frac{k}{m}} := \omega_0$ and call ω_0 the natural angular frequency. Notice that its units are radians per time. We also replace the linear combination coefficients c_1, c_2 by A, B. So, using the alternate letters, the general solution to

$$x'' + \omega_0^{\,2}\,x = 0$$

is

$$x(t) = A\cos\left(\omega_0\,t\right) + B\sin\left(\omega_0\,t\right).$$

This motion is called simple harmonic motion. The reason for this is that $x(t)$ can be rewritten as

$$x(t) = C\cos\left(\omega_0\,t - \alpha\right) = C\cos\left(\omega_0\left(t - \delta\right)\right)$$

in terms of an amplitude $C > 0$ and a phase angle α (or in terms of a time delay δ).