W is subspace if
α) closed under +
β) closed under scalar mult

Exercise 1) Most subsets of $\mathbb{R}^m$ are actually not subspaces. Show that

1a) $W = \{[x, y]^T \in \mathbb{R}^2 \ s.t.\ x^2 + y^2 = 4\}$ is not a subspace of $\mathbb{R}^2$.

1b) $W = \{[x, y]^T \in \mathbb{R}^2 \ s.t.\ y = 3x + 1\}$ is not a subspace of $\mathbb{R}^2$.

1c) $W = \{[x, y]^T \in \mathbb{R}^2 \ s.t.\ y = 3x\}$ is a subspace of $\mathbb{R}^2$. Then find a basis for this subspace.

1d) W = the solution space in $\mathbb{R}^n$ to a non-homogeneous matrix equation $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$ ($\underline{\mathbf{b}} \neq \underline{\mathbf{0}}$), with $A_{m\times n}$ is not a subspace. In particular, lines and planes that don't go through the origin are not subspaces.

1a) : W is not $\{\vec{0}\}$, line thru $\vec{0}$, all of $\mathbb{R}^2$. So W is not a subspace DONE

OR

$\begin{bmatrix}0\\2\end{bmatrix} \in W$. $\begin{bmatrix}2\\0\end{bmatrix} \in W$. $\begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$. $\begin{bmatrix}2\\2\end{bmatrix} \notin W$ because $2^2 + 2^2 = 8 \neq 4$. DONE

OR

$\begin{bmatrix}0\\2\end{bmatrix} \in W$. $29\begin{bmatrix}0\\2\end{bmatrix} \notin W$. $0^2 + 58^2 \neq 4$. DONE.

$0\begin{bmatrix}0\\2\end{bmatrix} \notin W$ $\quad 0^2 + 0^2 \neq 4$

1b). W is not $\{\vec{0}\}$, not a line thru $\vec{0}$, not $\mathbb{R}^2$. DONE

$\begin{bmatrix}0\\1\end{bmatrix} \in W$. $\begin{bmatrix}1\\4\end{bmatrix} \in W$. $\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}1\\5\end{bmatrix}$. $\notin W$. $\quad y = 3x + 1$? $\quad 5 \neq 3 + 1$ DONE

1c) W is a subspace : line thru $\vec{0}$.

$= \left\{\begin{bmatrix}x\\y\end{bmatrix} \text{ s.t. } y = 3x\right\}$. Let $\begin{bmatrix}x_1\\y_1\end{bmatrix}, \begin{bmatrix}x_2\\y_2\end{bmatrix} \in W$. ← (start here Wed)

$y_1 = 3x_1$
$y_2 = 3x_2$
$y_1 + y_2 = 3x_1 + 3x_2$
$y_1 + y_2 = 3(x_1 + x_2)$ → $y = 3x$

(α) $\begin{bmatrix}x_1\\y_1\end{bmatrix} + \begin{bmatrix}x_2\\y_2\end{bmatrix} = \begin{bmatrix}x_1 + x_2\\y_1 + y_2\end{bmatrix} \in W$.

(β) if $\begin{bmatrix}x_1\\y_1\end{bmatrix} \in W$, $y_1 = 3x_1$

$\Rightarrow cy_1 = c3x_1$

$cy_1 = 3\boxed{(cx_1)}$

so $\begin{bmatrix}\boxed{cx_1}\\cy_1\end{bmatrix} \in W$

better: $\begin{bmatrix}x\\3x\end{bmatrix} = x\begin{bmatrix}1\\3\end{bmatrix}$. $\quad W = span\left\{\begin{bmatrix}1\\3\end{bmatrix}\right\}$

1d) Solns $\vec{x}$ to $A\vec{x} = \vec{b}$ $\quad (\vec{b} \neq \vec{0})$

not a subspace.

$\vec{x}_1$ soln to $A\vec{x} = \vec{b}$ $\quad (A\vec{x}_1 = \vec{b})$

$\vec{x}_2$ " " " $\quad (A\vec{x}_2 = \vec{b})$.

α) check $A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = 2\vec{b}$.

solns not closed under +

Key facts about how subspaces DO arise:

There are two ways that subspaces arise: (These ideas will be important when we return to differential equations, in Chapter 5, although it's probably difficult to envision what they have to do with differential equations right now.)

1) $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n\}$.

Expressing a subspace this way is an explicit way to describe the subspace W, because you are "listing" all of the vectors in it. In this case we prefer that $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ be linearly independent, i.e. a basis, because that guarantees that each $\underline{\boldsymbol{w}} \in W$ is a unique linear combination of these spanning vectors.

Recall why W is a subspace: Let $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W \Rightarrow$

$$\left.\begin{aligned} \underline{\boldsymbol{v}} &= c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n \\ \underline{\boldsymbol{w}} &= d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n \end{aligned}\right\}$$

$$\Rightarrow \underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} = (c_1 + d_1)\underline{\boldsymbol{v}}_1 + (c_2 + d_2)\underline{\boldsymbol{v}}_2 + \ldots + (c_n + d_n)\underline{\boldsymbol{v}}_n \in W \quad \text{(verifies } \alpha\text{)}$$

and let $c \in \mathbb{R} \Rightarrow$

$$c\underline{\boldsymbol{v}} = cc_1\underline{\boldsymbol{v}}_1 + cc_2\underline{\boldsymbol{v}}_2 + \ldots + cc_n\underline{\boldsymbol{v}}_n \in W \quad \text{(verifies } \beta\text{)}$$

homogeneous matrix eqtn.

2) $W = \{\underline{\boldsymbol{x}} \in \mathbb{R}^n \text{ such that } A_{m \times n}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}\}$.

This is an implicit way to describe the subspace W because you're only specifying a homogeneous matrix equation that the vectors in W must satisfy, but you're not saying what the vectors are.

Why W is a subspace: Let $\underline{\boldsymbol{v}}, \underline{\boldsymbol{w}} \in W \Rightarrow$

$$A\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}},\ A\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}} \Rightarrow A\underline{\boldsymbol{v}} + A\underline{\boldsymbol{w}} = \underline{\boldsymbol{0}} \Rightarrow A(\underline{\boldsymbol{v}} + \underline{\boldsymbol{w}}) = \underline{\boldsymbol{0}} \Rightarrow \underline{\boldsymbol{v}} + \underline{\boldsymbol{w}} \in W \quad \text{(verifies } \alpha\text{)}$$

and let $c \in \mathbb{R} \Rightarrow$

$$A\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \Rightarrow cA\underline{\boldsymbol{v}} = c\underline{\boldsymbol{0}} = \underline{\boldsymbol{0}} \Rightarrow A(c\underline{\boldsymbol{v}}) = \underline{\boldsymbol{0}} \Rightarrow c\underline{\boldsymbol{v}} \in W \quad \text{(verfies } \beta\text{).}$$

....

Example: Last week we saw that

(1) $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$ with

$$W = span\left\{\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right\}$$

explicit way

is the collection of position vectors for points (x, y, z) satisfying the implicit equation

(2)

$$-2x - y + z = 0.$$

implicit way

In other words the explicit description of W as the span of two vectors in (1) corresponds to the implicit description of W as the space of solutions to the homogeneous matrix equation

$$\begin{bmatrix} -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}.$$

Tues Feb 28
Continuing our discussion of linear combinations, span, linear independence/dependence, subspace, basis, dimension.

4:30–6:00 LCB 218 study session

lin. comb. of $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is any $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$

$span\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n,\ c_j \in \mathbb{R}\}$ (all linear combos)

$\{\vec{v}_1, \vec{v}_2 \dots \vec{v}_n\}$ linearly indep: $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0} \Longrightarrow c_1 = c_2 = \dots = c_n = 0$

W subspace means: W closed under addition (α), and closed under scalar multiplication (β)

basis for a subspace W is set of vectors in W that span W & are linearly independent

dimension of subspace is # of vectors in a basis

Exercise 1) Use geometric reasoning to argue why the only subspaces of $\mathbb{R}^2$ are

- (0) The single vector $[0, 0]^T$, or ⟵ $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$ is a very small subspace W
- (1) A line through the origin, i.e. $span\{\boldsymbol{u}\}$ for some non-zero vector $\boldsymbol{u}$, or
- (2) All of $\mathbb{R}^2$.

α) $\vec{0} + \vec{0} = \vec{0} \in W$
β) $c\vec{0} = \vec{0} \in W$
(Note: $\vec{0} \in W$ whenever W is a subspace)

(1) If W contains more than $\vec{0}$
Let $\vec{u} \in W$, $\vec{u} \neq \vec{0}$. (β) $\Longrightarrow \{t\vec{u},\ t \in \mathbb{R}\} \subset W$ is a line thru $\vec{0}$

(2) If $span\{\vec{u}\}$ is not all of W, pick $\vec{v} \in W$, $\vec{v} \notin span\,\vec{u}$.
(α), (β) $\Longrightarrow \underbrace{\{t\vec{u} + s\vec{v},\ s,t \in \mathbb{R}\}}_{\mathbb{R}^2} \subset W$

Exercise 2) Use matrix theory to show that the only subspaces of $\mathbb{R}^3$ are

- (0) The single vector $[0, 0, 0]^T$, or
- (1) A line through the origin, i.e. $span\{\boldsymbol{u}\}$ for some non-zero vector $\boldsymbol{u}$, or
- (2) A plane through the origin, i.e. $span\{\boldsymbol{u}, \boldsymbol{v}\}$ where $\boldsymbol{u}$, $\boldsymbol{v}$ are linearly independent, or
- (3) All of $\mathbb{R}^3$.

Exercise 3) What are the dimensions of the subspaces in Exercise ~~4~~ 1 and Exercise ~~3~~ 2? How do these ideas generalize to subspaces of $\mathbb{R}^n$?

$\in$ is an element of
$\subset$ is contained in

② Let W be a subspace of $\mathbb{R}^3$.

(0) $\vec{0} \in W$ because if $\vec{w} \in W$, then (β) $\Rightarrow 0\vec{w} \in W$, $0\vec{w} = \vec{0}$

(1) if more, let $\vec{u} \in W$, $\vec{u} \neq \vec{0}$
(β) $\Longrightarrow \{t\vec{u},\ t \in \mathbb{R}\} \subset W$. line thru $\vec{0}$.
$\{t\vec{u},\ t \in \mathbb{R}\} = span\{\vec{u}\}$.

(2) if more, let $\vec{v} \in W$, $\vec{v} \notin span\,\vec{u}$.
(β) $\Longrightarrow \{s\vec{v},\ s \in \mathbb{R}\} \subset W$
(α) $\Longrightarrow \{t\vec{u} + s\vec{v},\ s,t \in \mathbb{R}\} \subset W$
$\{t\vec{u} + s\vec{v},\ s,t \in \mathbb{R}\} = span\{\vec{u}, \vec{v}\}$ plane thru $\vec{0}$

$$\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \cancel{\begin{bmatrix} 1 & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

(3) if more, pick $\vec{w} \in W$, $\vec{w} \notin span\{\vec{u}, \vec{v}\}$
(α), (β) $\Longrightarrow \{t\vec{u} + s\vec{v} + p\vec{w}\} \subset W$
$\{t\vec{u} + s\vec{v} + p\vec{w}\} = \mathbb{R}^3$

$$\left[\begin{array}{ccc|c} \vec{u} & \vec{v} & \vec{w} & \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right] \quad \cancel{\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}}$$

$\vec{w} \in span\{\vec{u}, \vec{v}\}$

Usually in applications we do not start with a basis for a subspace - rather this is the goal we search for, since the entire subspace may be reconstructed explicitly and precisely from the basis (which is why a basis is called "a basis"). Usually, our subspace W in $\mathbb{R}^m$ is likely to be described in an implicit manner, as the solution space to a homogeneous matrix equation. In Chapter 5 the subspaces W will be the solution spaces to "homogeneous linear differential equations."

If we wish to find a basis for the homogeneous solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^n \text{ such that } A_{m \times n}\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$, then the following algorithm will always work: reduce the augmented matrix, backsolve and write the explicit solution in linear combination form. The vectors that you are taking linear combinations of will always span the solution space, by construction. If you follow this algorithm they will automatically be linearly independent, so they will be a basis for the solution space. This is illustrated in the large example below:

Exercise 4 Consider the matrix equation $A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}$, with the matrix A (and its reduced row echelon form) shown below:

$$A := \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{matrix} \\ \left[\begin{matrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 2 & 4 & 1 & 4 & 1 & 7 \\ -1 & -2 & 1 & 1 & -2 & 1 \\ -2 & -4 & 0 & -2 & -2 & -4 \end{matrix}\right|\left.\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right] \end{array} \rightarrow \left[\begin{matrix} 1 & 2 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right|\left.\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right]$$

Find a basis for the solution space $W = \{\underline{\mathbf{x}} \in \mathbb{R}^6 \text{ s.t. } A\,\underline{\mathbf{x}} = \underline{\mathbf{0}}\}$ by backsolving, writing your explicit solutions in linear combination form, and extracting a basis. Explain why these vectors span the solution space and verify that they're linearly independent.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2t_2 - t_4 - t_5 - 2t_6 \\ t_2 \\ -2t_4 + t_5 - 3t_6 \\ t_4 \in \mathbb{R} \\ t_5 \in \mathbb{R} \\ t_6 \in \mathbb{R} \end{bmatrix} = t_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_5\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_6\begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \overset{\text{if}}{=} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$t_2 = 0$. $t_4 = 0$ $t_5 = 0$ $t_6 = 0$

$$W = \operatorname{span}\left\{ \underset{\vec{z}_1}{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{z}_2}{\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\vec{z}_3}{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\vec{z}_4}{\begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

claim $\{\vec{z}_1, \vec{z}_2, \vec{z}_3, \vec{z}_4\}$ is a basis for W

(1) is their span all of W? we just showed that!
(2) are they linearly ind.? • call the c's, "t"'s
Yes

Solution (don't peek :-): backsolve, realizing that the augmented matrices have final columns of zero. Label the free variables with the letter "t", and subscripts to show the non-leading 1 columns from which they arose:

$$x_6 = t_6, x_5 = t_5, x_4 = t_4, x_3 = -2\,t_4 + t_5 - 3\,t_6$$
$$x_2 = t_2, x_1 = -2\,t_2 - t_4 - t_5 - 2\,t_6\,.$$

In vector form and then linear combination form this is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix} = t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the four vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for the solution space:

- *They span the solution space by construction.*
- *They are linearly independent because if we set a linear combination equal to zero:*

$$t_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_5 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_6 \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2\,t_2 - t_4 - t_5 - 2\,t_6 \\ t_2 \\ -2\,t_4 + t_5 - 3\,t_6 \\ t_4 \\ t_5 \\ t_6 \end{bmatrix}$$

then looking in the second entry implies $t_2 = 0$, the fourth entry implies $t_4 = 0$, and similarly $t_5 = t_6 = 0$.

Math 2250-004
Wed Mar 1

- Wed: Mostly Tuesday's notes
- Quiz day!
- Lab tomorrow → function spaces as vector spaces

5.1 Second order linear differential equations, and vector space theory connections.

Definition: A $\boxed{\text{vector space}}$ is a collection of objects together with and "addition" operation "+", and a scalar multiplication operation, so that the rules below all hold.

- (α) Whenever $f, g \in V$ then $f + g \in V$. (closure with respect to addition)
- (β) Whenever $f \in V$ and $c \in \mathbb{R}$, then $c \cdot f \in V$. (closure with respect to scalar multiplication)

$\Big\}$ subspace properties

$(f+g)' = f' + g'$
$(cf)' = cf'$
$\cancel{(fg)' = f'g + fg'}$

As well as:
(a) $f + g = g + f$ (commutative property)
(b) $f + (g + h) = (f + g) + h$ (associative property)
(c) $\exists\ 0 \in V$ so that $f + 0 = f$ is always true. →
(d) $\forall\ f \in V\ \exists\ -f \in V$ so that $f + (-f) = 0$ (additive inverses)
(e) $c \cdot (f + g) = c \cdot f + c \cdot g$ (scalar multiplication distributes over vector addition)
(f) $\left(c_1 + c_2\right) \cdot f = c_1 \cdot f + c_2 \cdot f$ (scalar addition distributes over scalar multiplication)
(g) $c_1 \cdot \left(c_2 \cdot f\right) = \left(c_1 c_2\right) \cdot f$ (associative property)
(h) $1 \cdot f = f,\ (-1) \cdot f = -f,\ 0 \cdot f = 0$ (these last two actually follow from the others).

Examples we've seen:
(1) $\mathbb{R}^m$, with the usual vector addition and scalar multiplication, defined component-wise
(2) subspaces W of $\mathbb{R}^m$, which satisfy (α),(β), and therefore automatically satisfy (a)-(h), because the vectors in W also lie in $\mathbb{R}^m$.

Exercise 0) In Chapter 5 we focus on the <u>vector space</u>
$$V = C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R}\ s.t.\ f\ \textit{is a continuous function}\}$$
and its subspaces. Verify that the vector space axioms for linear combinations are satisfied for this space of functions. Recall that the function $f + g$ is defined by $(f + g)(x) := f(x) + g(x)$ and the scalar multiple $c f(x)$ is defined by $(c f)(x) := c f(x)$. What is the zero vector for functions?

Because the vector space axioms are exactly the arithmetic rules we used to work with linear combination equations, all of the concepts and vector space theorems we talked about for $\mathbb{R}^m$ and its subspaces make sense for the function vector space V and its subspaces. In particular we can talk about

- the span of a finite collection of functions $f_1, f_2, \dots f_n$.
- linear independence/dependence for a collection of functions $f_1, f_2, \dots f_n$.
- subspaces of V
- bases and dimension for finite dimensional subspaces. (The function space V itself is infinite dimensional, meaning that no finite collection of functions spans it.)

Example $\quad f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2$

$$\operatorname{span}\{f_1, f_2, f_3\} = \{af_1 + bf_2 + cf_3 \ , \ a, b, c \in \mathbb{R}\}$$
$$= \{f(x) = a \cdot 1 + b \cdot x + c \cdot x^2, \ a, b, c \in \mathbb{R}\}$$
$$= \text{space of polynomials of degree} \leq 2.$$

are f_1, f_2, f_3 linearly ind.

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$$ ← zero function

at each x $\quad \cancel{c_1} + c_2 x + c_3 x^2 = 0$

@ $x = 0$: $\quad c_1 = 0$

@ $x = 1$: $\quad c_2 + c_3 = 0$

@ $x = -1$: $\quad -c_2 + c_3 = 0$

$\Longrightarrow c_2, c_3 = 0$

so $\{1, x, x^2\}$ is linearly independent.