Prof. Korevaar

Math 2250-004 Week 1 notes
We will not necessarily finish the material from a given day's notes on that day. We may also add or subtract some material as the week progresses, but these notes represent an in-depth outline of what we will cover. These notes are for sections 1.1-1.3, and part of 1.4.

Monday January 9

• Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2250spring17

§1.1 & 1.2

• Note that there is a quiz this Wednesday on the material we cover today and tomorrow, and that your first lab meeting is this Thursday. Your first homework assignment will be due next Wednesday, January 17.

Then, let's begin!

• What is an n^{th} order differential equation (DE)?

any equation involving a function $y = y(x)$ and its derivatives, for which the highest derivative appearing in the equation is the n^{th} one, $y^{(n)}(x)$; i.e. any equation which can be written as

$$F\left(x, y(x), y'(x), y''(x), \ldots y^{(n)}(x)\right) = 0.$$

Exercise 1: Which of the following are differential equations? For each DE determine the order.

a) For $y = y(x)$, $\left(y''(x)\right)^2 + \sin(y(x)) = 0$

b) For $x = x(t)$, $x'(t) = 3\,x(t)(10 - x(t))$.

c) For $x = x(t)$, $x' = 3\,x\,(10 - x)$.

d) For $z = z(r)$, $z'''(r) + 4\,z(r)$.

e) For $y = y(x)$, $y' = y^2$.

yes! 2nd order

yes! 1st order

yes! same as (b)! abbreviated (don't get fooled)

No! no = sign! not an equation
$z'''(r) + 4z(r) = \sin(r) + 7$ is a DE (3rd order)

YES. order 1st

A solution function $y(x)$ to the differential equation $F\left(x, y, y', y'', y^{(n)}\right) = 0$ defined on some interval I is any function $y(x)$ which makes the differential equation a true equality for all x in I.

sol's are fcns, not numbers!

A solution function $y(x)$ to a first order differential equation $F(x, y, y') = 0$ on the interval I which also satisifies $y\left(x_0\right) = y_0$ for a specified $x_0 \in I$ and $y_0 \in \mathbb{R}$ is called a solution to the initial value problem (IVP)

$$\text{IVP} \quad \begin{cases} F(x, y, y') = 0 \\ y\left(x_0\right) = y_0 \end{cases}$$

· Initial condition

Exercise 2: Consider the differential equation $\dfrac{dy}{dx} = y^2$ from (1e).

2a) Show that functions $y(x) = \dfrac{1}{C - x}$ solve the DE (on any interval not containing the constant C).

2b) Find the appropriate value of C to solve the initial value problem

$$y' = y^2$$
$$y(1) = 2\,.$$

LHS RHS

2a) $\dfrac{dy}{dx} = (y(x))^2$

are $y = \dfrac{1}{C-x}$ soln's (C is any constant)?

plug into DE, & see if we get a true identity.

$y = (C-x)^{-1}$

LHS $y' = -1\,(C-x)^{-2}(-1) = (C-x)^{-2} = \dfrac{1}{(C-x)^2}$

RHS $y^2 = \left(\dfrac{1}{C-x}\right)^2$

$y' = y^2$? yes

LHS = RHS for these fcns $y(x)$, so they are solutions to the DE.

2b) Solve IVP:

$y = \dfrac{1}{C-x}$

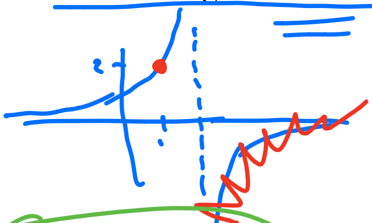
$y(1) = 2$

$y(1) = \dfrac{1}{C-1} = 2 \Rightarrow C - 1 = \frac{1}{2} \Rightarrow C = 3/2$

soln $\boxed{y(x) = \dfrac{1}{3/2 - x}}$

$y(1) = \dfrac{1}{3/2 - 1} = \dfrac{1}{1/2} = 2$ ✓

2c) What is the largest interval on which your solution to (b) is defined as a differentiable function? Why?



$y(x) = \dfrac{1}{3/2 - x}$

$y(1) = 2$

interval to contain $x_0 = 1$

$-\infty < x < 3/2$

2d) Do you expect that there are any other solutions to the IVP in 2b? Hint: The graph of the IVP solution function we found is superimposed onto a "slope field" below: The line segments at points (x, y) have values y^2, because solutions graphs to the differential equation

$$\boxed{y' = y^2}$$

will have slopes given by the derivatives of the solutions $y(x)$. This might give you some intuition about whether you expect more than one solution to the IVP.

the graph of any soln $y = y(x)$,

(1) has slope y' at any pt (x, y) on graph

(2) but because $y(x)$ solves the DE, the slope is also y^2

No : (I misread the question in class)

~~Yes~~
- know where we start
- know slope where-ever we are
- should only be one graph

Expect unique solutions

$y' = y^2$

$y = \dfrac{1}{3/2 - x}$

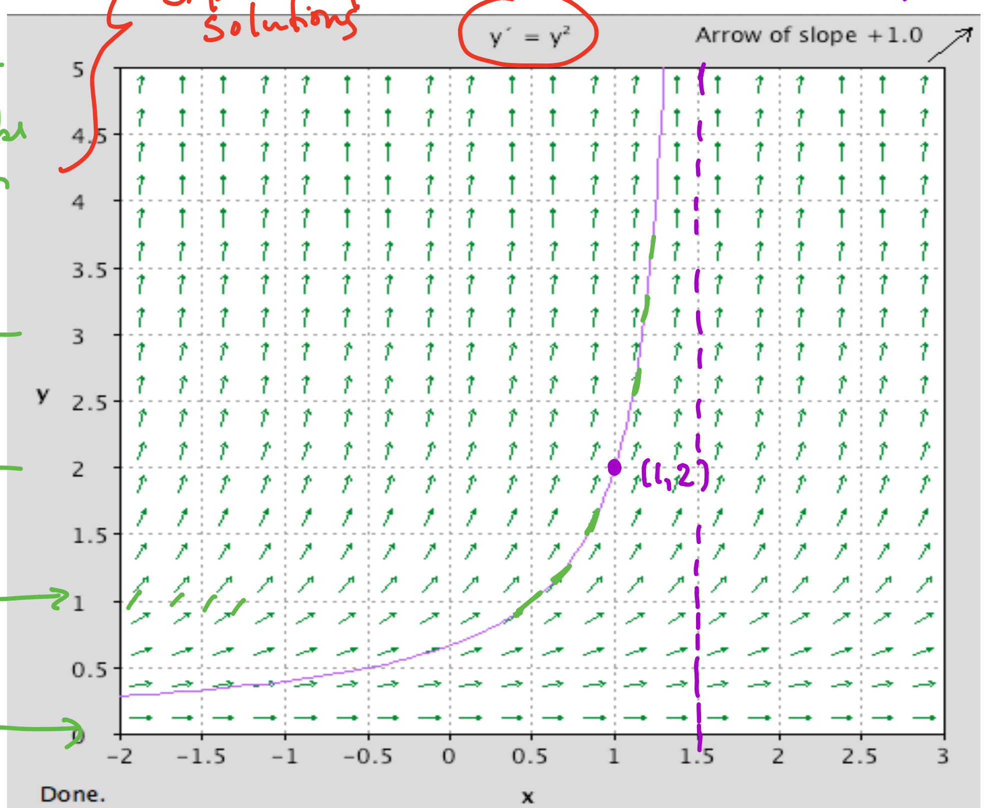
$y(1) = 2$

slopes 9 along $y=3$

$y=2$ slopes 4

$y=1$ slopes $y^2=1$

green slopes $=0$ along $y=0$



• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.

In fact, you've encountered differential equations in previous mathematics and/or physics classes:

• 1^{st} order differential equations: rate of change of function depends in some way on the function value, the variable value, and nothing else. For example, you've studied the population growth/decay differential equation for $P = P(t)$, and k a constant, given by

$$P'(t) = k\,P(t)$$

and having applications in biology, physics, finance.

• 2^{nd} order DE's: Newton's second law (change in momentum equals net forces) often leads to second order differential equations for particle position functions $x = x(t)$ in physics.

$m\,x''(t) = \text{net forces} = \text{expression } x, x', t$

(e.g. $m x'' = -mg$)

Exercise 3: The mathematical model in which the time rate of change of a population $P(t)$ is proportional to that population is expressed mathematically as

$P'(t)$ $=$ $k\cdot$

$$\boxed{\frac{dP}{dt}} = kP$$

$P'(t) = k\,P(t)$

where k is the proportionality constant.

3a) Find all solutions to this differential equation by using the chain rule backwards.

3b) The method of "separation of variables" is taught in most Calc I courses, and we'll cover it in detail in section 1.4. It's an algorithm which hides the "chain rule backwards" technique by treating the derivative $\frac{dP}{dt}$ as a quotient of differentials. Recall this magic algorithm to recover the solutions from (3a).

trick: 3a). Solve the DE $P'(t) = k\,P(t)$

$\Rightarrow \frac{P'(t)}{P(t)} = k$

$\Rightarrow \int \frac{P'(t)}{P(t)}\,dt = \int k\,dt$

u-sub.
$u = P(t)$
$du = P'(t)\,dt$
$\int \frac{du}{u} = \ln|u| + C_1$

$\ln|P(t)| + C_1 = kt + C_2$

$\ln|P(t)| = kt + C_3$ $\quad (C_3 = C_2 - C_1)$

$e \qquad e$

$|P(t)| = e^{kt}e^{C_3}$

$\Rightarrow P(t) = \pm e^{kt}e^{C_3}.$

$\Rightarrow \boxed{P(t) = Ce^{kt}}$ $\quad (C = \pm e^{C_3})$

usually write $P(t) = P_0 e^{kt}$

because $P(0) = Ce^0 = C$, & P_0 is what we call $P(0)$

$$\boxed{P(t) = P_0 e^{kt}}$$

shortcut! (works for separable DE's).

$\frac{dP}{dt} = kP$

$\int \frac{dP}{P} = \int k\,dt$

$\ln|P| = kt + C$

SAME

Exercise 4) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate proportional to to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use this model to derive the differential equation

$$\frac{dT}{dt} = -k(T - A).$$

✓ ?

$T'(t) = k_1(T - A)$

• if $T > A$, $T'(t) < 0$, $T - A > 0$ $\Rightarrow k_1 < 0$

if $T < A$, $T'(t) > 0$, $T - A < 0$, $k_1 < 0$

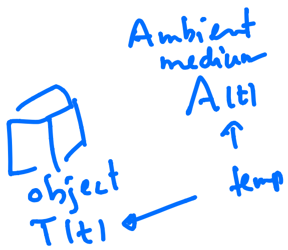
b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead? Yes

c) Use this model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70° F. An hour later the body temperature has decreased to 60°. It's been a winter inversion in SLC, with constant ambient temperature 30°. Assuming the Newton's law model, estimate the time of death.

① model, DE
② DE, solve
③ apply.

c)

so $k_1 < 0$
so let's call it "$-k$" instead, $k > 0$
(we like positive constants)



solve $\frac{dT}{dt} = -k(T - A)$ const

$$\int \frac{dT}{T-A} = \int -k\,dt$$

$$e^{\ln|T-A|} = e^{-kt + C}$$

$$|T - A| = e^{-kt}e^{C}$$

$$T - A = Ce^{-kt}$$

$$\boxed{T(t) = A + Ce^{-kt}}$$

c): set $t=0$ to be 3:00 p.m. use hour units for time

$T(0) = 70$
$T(1) = 60$
$A = 30$

$$T(t) = 30 + Ce^{-kt}$$

• find C: $T(0) = 70 = 30 + C \Rightarrow C = 40$
• find k: $T(1) = 60$ $\quad 60 = 30 + 40e^{-k}$ $\quad k = .2876$
• set $T(t) = 98.6$ & solve for t.

$$T(t) = 30 + 40e^{-kt}$$

$$98.6 = 30 + 40e^{-kt}$$

solve for t: $t = -1.875$ hours.
($t=0$ was 3:00 p.m.)

$$\begin{array}{r} 3 \\ -1.875 \\ \hline 1.125 \end{array}$$ o'clock

$\approx 1:07$

$$\begin{array}{r} .125 \\ 60 \\ \hline \cancel{7.5}\ 7.50 \end{array}$$

Math 2250-004
Tues Jan 10

Recall that course info - syllabus, class notes, homework, etc. is posted at our web page
http://www.math.utah.edu/~korevaar/2250spring17
There will also be course material posted on our CANVAS page.

• Recall from Monday that a 1^{st} order DE is an equation involving a function and its first derivative. We may chose to write the function and variable as $y = y(x)$. In this case the differential equation is an equation equivalent to one of the form

$$F(x, y, y') = 0 .$$

• We can often use algebra to solve for y', to get what we call the **standard form** for a first order DE:

$$y' = f(x, y) .$$

• If we want our solution function to a DE to also satisfy $y(x_0) = y_0$, and if our DE is written in standard form, then we say that we are solving an **initial value problem** (IVP):

$$\text{IVP} \left\{ \begin{array}{ll} y' = f(x, y) & \longleftarrow \text{DE} \\ y(x_0) = y_0 . & \longleftarrow \text{IC} \end{array} \right.$$

With these ideas in mind, let's finish Monday's notes, including Exercises 3 and 4 (assuming we didn't finish them on Monday).

Tuesday notes on Wed.

Section 1.2: differential equations equivalent to ones of the form

$$y'(x) = f(x)$$

$\frac{dy}{dx} = f(x,y)$ in general

easiest case: what fcns $y(x)$ have derivatives $f(x)$? ans: antiderivatives!

which we solve by direct antidifferentiation

$$y(x) = \int f(x)\, dx = F(x) + C.$$

Exercise 1 Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2 + 4}$$

$$y(0) = 0$$

(a) solve the DE

$$y = \int x\sqrt{x^2+4}\, dx$$

$u = x^2 + 4$

$du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

$$= \int u^{1/2} \tfrac{1}{2} du$$

$$= \frac{1}{2}\int u^{1/2} du$$

$$= \frac{1}{2}\cdot\frac{2}{3} u^{3/2} + C$$

$$= \frac{1}{3} u^{3/2} + C$$

$$\boxed{y = \frac{1}{3}(x^2+4)^{3/2} + C}$$

(b) IVP:

$$y(0) = 0 = \frac{1}{3}\cdot 4^{3/2} + C$$

$$0 = \frac{1}{3}8 + C$$

$$-8/3 = C$$

IVP $\boxed{y = \frac{1}{3}(x^2+4)^{3/2} - 8/3}$

An important class of such problems arises in physics usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

$m\,x''(t) = $ net forces
if
$= f(t)$ then §1.2

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

Exercise 2:
a) If the units for position are meters m and the units for time are seconds s, what are the units for velocity and acceleration? (These are *mks* units.) vel. units m/s ; accel m/s^2
b) Same question, if we use the English system in which length is measure in feet and time in seconds. Could you convert between *mks* units and English units?

velocity units ft/sec
accel. unit ft/sec^2

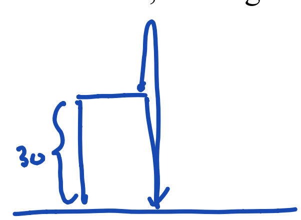
units $x(t)$ length

$$v(t) = x'(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \leftarrow \frac{\text{length}}{\text{time}}$$

$$v'(t) = x''(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t} \quad \frac{\text{length}/\text{time}}{\text{time}} = \frac{\text{length}}{\text{time}^2}$$

Exercise 3: A projectile with very low air resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof.

a) Neglecting friction, how high will the object get above ground?
b) When does the object land ?



(find formulas for ht & velocity).

Let $y(t)$ be height @ time t (m)
$y(0) = 30$, (choice to set ground level as $y=0$)
$v(0) = y'(0) = 50$ m/s

$m\,y''(t) = -mg$ $\quad g \approx 9.8\ m/s^2$

$\boxed{y''(t) = -g}$ $\quad$ Let's Ex 4 at this point

transpose Ex 4:

$\boxed{y(t) = -\frac{1}{2}g t^2 + v_0 t + y_0}$

$\boxed{v(t) = -gt + v_0}$

$\boxed{\begin{aligned} y(t) &= -4.9t^2 + 50t + 30 \\ v(t) &= -9.8t + 50 \end{aligned}}$

a) how high?
set $y'(t) = 0$
$v(t) = 0$
solve for t (when)

$-9.8t + 50 = 0$
$\boxed{t = 5.1}$ sec
height $\boxed{y(5.1) = 157.55\ m}$

b) When does it land.
Solve $y(t) = 0$ for t
$\boxed{t = 10.76 \text{ sec}}$

Exercise 4:

Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a\,.$$

(This could happen for vertical motion, e.g. near the earth's surface with $a = g \approx 9.8\ \frac{m}{s^2} \approx 32\ \frac{ft}{s^2}$, as well as in other situations.)

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{max} = \frac{1}{2}\frac{v_0^2}{a}\,.$$

(This formula may help with some homework or lab problems, besides being interesting.) *

a)

$$x''(t) = -a$$

$$x'(t) = \int -a\,dt = -at + C$$

$$v(t) = -at + C$$

@ $t=0$: $v(0) = 0 + C \Rightarrow C = v_0$

$$\int x'(t) = \boxed{v(t) = -at + v_0}$$

$$\Rightarrow x(t) = \int -at + v_0\,dt$$

$$x(t) = -\tfrac{1}{2}at^2 + v_0 t + C$$

@ $t=0$: $x_0 = 0 + 0 + C. \Rightarrow x_0 = C$

$$\boxed{x(t) = -\tfrac{1}{2}at^2 + v_0 t + x_0}$$

Here's another fun example from section 1.2, which also reviews important ideas from Calculus - in particular we will see how the fact that the slope of a graph $y = g(x)$ is the derivative $\frac{dy}{dx}$ can lead to first order differential equations.

Exercise 5: (See text, page 16). A swimmer wishes to cross a river of width $w = 2\,a$, by swimming directly towards the opposite side, with constant transverse velocity v_S. The river velocity is fastest in the middle and is given by an even function of x, for $-a \le x \le a$. The velocity equal to zero at the river banks. For example, it could be that

$$\boxed{v_R(x) = v_0\left(1 - \frac{x^2}{a^2}\right)}.$$

See the configuration sketches below.

a) Writing the swimmer location at time t as $(x(t), y(t))$, translate the information above into expressions for $x'(t)$ and $y'(t)$.

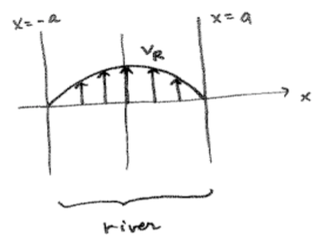
b) The parametric curve describing the swimmer's location can also be expressed as the graph of a function $y = y(x)$. Show that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right).$$

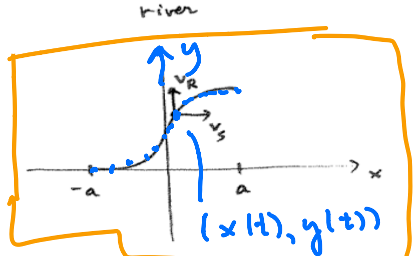
Filled in after class on Wed

c) Compute an integral or solve a DE, to figure out how far downstream the swimmer will be when she reaches the far side of the river.

set up: a)
$$x'(t) = v_s \quad \text{const}$$
$$y'(t) = v_R(x(t)) = v_0\left(1 - \frac{x^2}{a^2}\right)$$



b) the parametric curve of locations at time t, i.e. $t \to (x(t), y(t))$ can also be thought of as a graph $y = y(x)$ (where $x = x(t)$) so by Calc 1,



from (a)

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \Rightarrow \underbrace{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_0(1 - x^2/a^2)}{v_s} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)}_{\frac{dy}{dx} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)\ ✓}$$

c) from orange configuration, we want $y(a) - y(-a)$ where $y(x)$ solves IVP $\begin{cases} \frac{dy}{dx} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right) \\ y(-a) = 0 \leftarrow \text{our choice of origin on y-axis} \end{cases}$

Could solve IVP at right, or just use Fund. Theorem of Calc:

$$y(a) - y(-a) = \int_{-a}^{a} y'(x)\,dx = \int_{-a}^{a} \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 2\int_0^a \frac{v_0}{v_s}\left(1-\frac{x^2}{a^2}\right)dx$$ (integrand is even function)

$$= \frac{2v_0}{v_s}\left[x-\frac{x^3}{3a^2}\right]_0^a = \frac{2v_0}{v_s}\Big[\underbrace{a-\frac{a}{3}}_{\frac{2}{3}a}\Big]$$

$$\boxed{=\frac{4v_0 a}{3v_s}}$$ ← final answer

Math 2250-004
Wed Jan 11

- Quiz today at end of class, on section 1.1-1.2 material

- After finishing Tuesday's notes ~~if~~ is necessary, begin Section 1.3: slope fields and graphs of differential equation solutions: Consider the first order DE IVP for a function $y(x)$:

$$\boxed{y' = f(x,y)\ ,\ y(x_0) = y_0\,.}$$

for soln $y(x)$ its graph $y=y(x)$
$y'(x)$ is slope of graph, slope also given by $f(x,y)$

If $y(x)$ is a solution to this IVP and if we consider its graph $y = y(x)$, then the IC means the graph must pass through the point (x_0, y_0). The DE means that at every point (x,y) on the graph the slope of the graph must be $f(x,y)$. (So we often call $f(x,y)$ the "slope function" for the differential equation.) This gives a way of understanding the graph of the solution $y(x)$ even without ever actually finding a formula for $y(x)$! Consider a **slope field** near the point (x_0, y_0): at each nearby point (x,y), assign the slope given by $f(x,y)$. You can represent a slope field in a picture by using small line segments placed at representative points (x,y), with the line segments having slopes $f(x,y)$.

Exercise 1: Consider the differential equation $\dfrac{dy}{dx} = x - 3$, and then the IVP with $\boxed{y(1) = 2}$.
a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \le x \le 5$, $0 \le y \le 6$. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.
b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.
c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).

a) $\dfrac{dy}{dx} = x-3$

sketch slope field.
§1.2 ex $f(x,y) = f(x)$
slope only depends on x

x	slope fcn $x-3$
$x=3$	0
4	1
2	-1
1	-2
0	-3
5	2
2.5	$-.5$

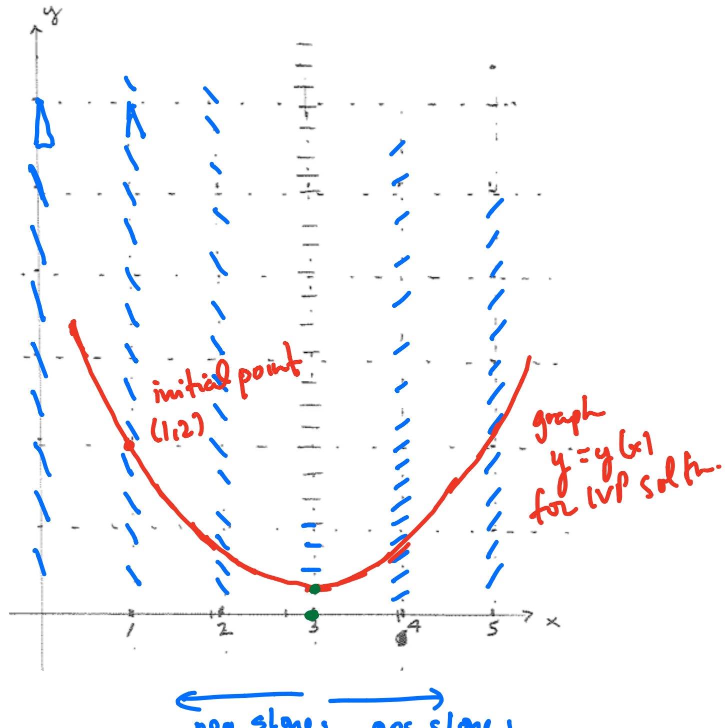
b) $\dfrac{dy}{dx} = x-3$
$y(1) = 2$

c) $y'(x) = x-3$

$y(x) = \int x-3\,dx$

$y = \dfrac{x^2}{2} - 3x + C$

$y(1)=2$:
$2 = \frac{1}{2} - 3 + C$
$2 = -2\tfrac{1}{2} + C$
$\frac{9}{2} = 4\tfrac{1}{2} = C$



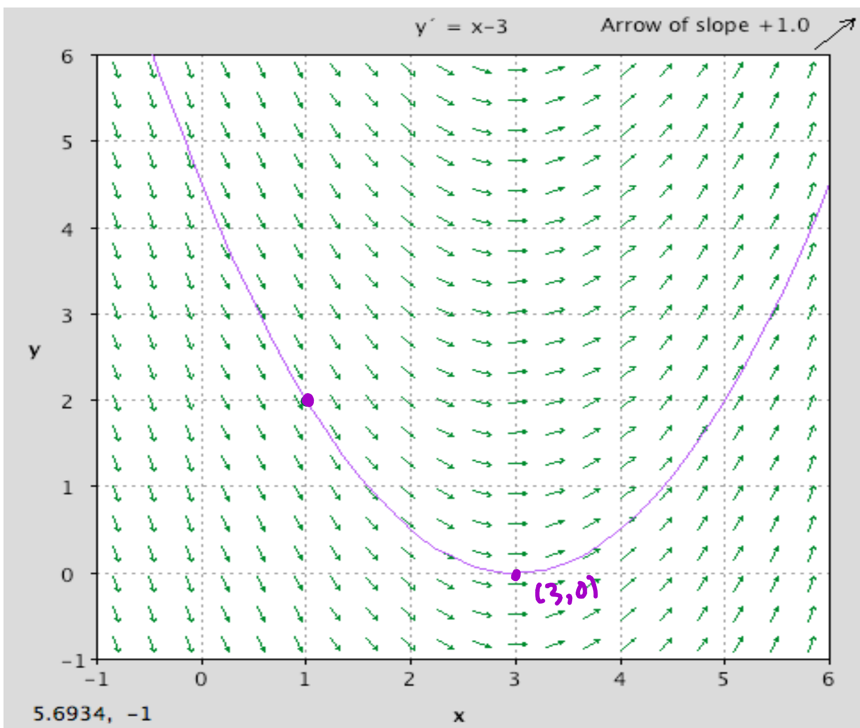
$y = \dfrac{x^2}{2} - 3x + \dfrac{9}{2}$
$= \frac{1}{2}(x^2 - 6x + 9) = \frac{1}{2}(x-3)^2$. Soln graph is $\underline{y = \frac{1}{2}(x-3)^2}$ vertex $(3,0)$

The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

$$\frac{\triangle y}{\triangle x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in *x* and *y*, starting from the initial point $\left(x_0, y_0\right)$. In this way one can approximate solution functions to IVPs, and their graphs. You can find an applet to do this by googling "dfield" (stands for "direction field", which is a synonym for slope field). Here's a picture like the one we sketched by hand on the previous page. The solution graph was approximated using numerical ideas as above, and this numerical technique works for much more complicated differential equations, e.g. when solutions exist but don't have closed form formulas. The program "dfield" was originally written for Matlab, and you can download a version to run inside that package. Or, you can download stand-alone java code.

google dfield
download java applet



Exercise 2: Consider the IVP

$y(x) = x+1+Ce^x$

LHS $y' = 1 + Ce^x$

$$\frac{dy}{dx} = y - x$$

RHS $y - x = (\cancel{x}+1+Ce^x) - \cancel{x}$

$$y(0) = 0$$

LHS = RHS, so $y(x)$ is a soln.

insert here.

a) Check that $\boxed{y(x) = x + 1 + C\,e^x}$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

b) Solve the IVP by choosing appropriate C.

insert here

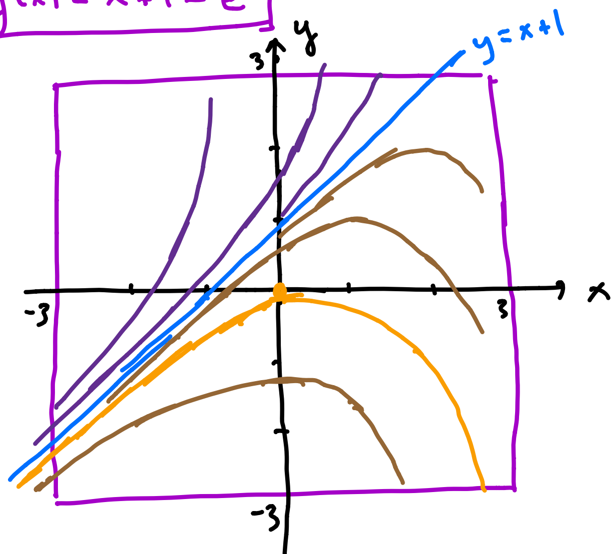
$y(0) = 0+1+C \overset{\text{want}}{=} 0 \Rightarrow C=-1$

c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

d) Compare your work in (c) with the picture created by dfield on the next page.

b) soln $\boxed{y(x) = x+1-e^x}$

c)

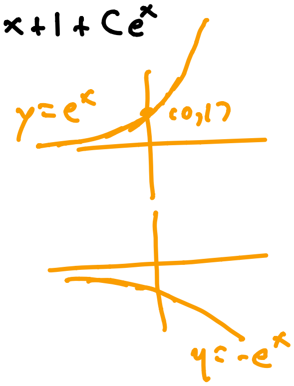


$y(x) = x+1+Ce^x$

$C<0$

$C>0$

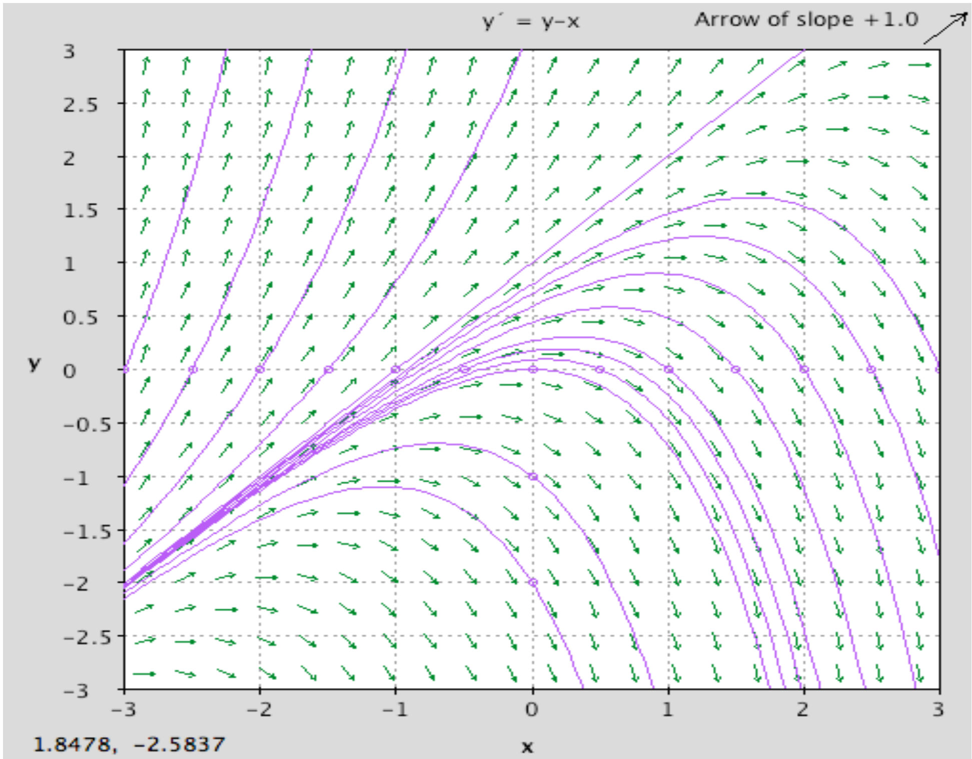
$y(x) = x+1+Ce^x$



$\lim_{x\to-\infty} e^x = 0$

$y' = 1-e^x$

$y'' = -e^x$



y´ = y-x
Arrow of slope +1.0
y
x
1.8478, -2.5837

• We'll use Wednesday's notes for a lot of the class

Math 2250-004
Fri Jan 13

• I'll post a copy of the lab in our HW page
• our CANVAS page exists
• next week's notes should be posted by 3:00 today. Print these yourself for Tuesday class

1.3-1.4: slope fields; existence and uniqueness for solutions to IVPs; examples we can check with separation of variables.

Exercise 1: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 .$$

not §1.2 problem! $y'(x) = f(x)$, $y(x) = \int f(x)dx$

a) Use separation of variables to find solutions to this DE...the "magic" algorithm that we talked about at the start of the week, but didn't explain the reasoning for. It is de-mystified on the next page of today's notes.
b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a? (You can sketch by hand, I'll use "dfield" on my browser.)
c) Explain why each IVP has a solution, but this solution does not exist for all x.

You can download the java applet "dfield" from the URL
http://math.rice.edu/~dfield/dfpp.html
(You also have to download a toolkit, following the directions there.)

shorthand for $y'(x) = 1 + y(x)^2$

BAD: $\frac{dy}{dx} = 1 + y^2$

$\int \frac{dy}{dx}\, dx = \int 1 + y^2\, dx$ ($y(x)$)

$y = x + ??$

don't know $y(x)$
don't know $y(x)^2$
so you can't compute $\int y(x)^2 dx$

RIGHT WAY:

$\frac{1}{1+y^2}\frac{dy}{dx} = 1$

$\int \frac{dy}{1+y^2} = \int dx$!

magic

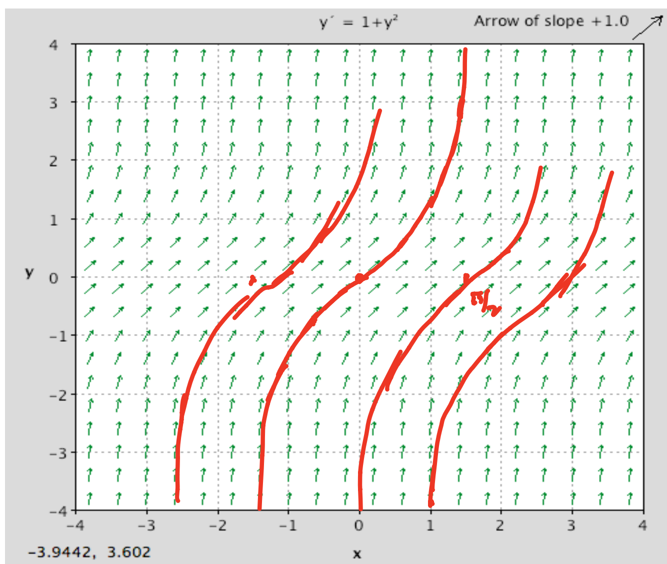
$\arctan(y) = x + C$

$\tan(\arctan(y)) = \tan(x+C)$

$y = \tan(x+C)$

e.g. $C=0$, $y = \tan x$ — vert. asym @ $x = \pm\pi/2$

solve IVP for $\begin{cases} y' = 1+y^2 \\ y(0) = 0 \end{cases}$



$y = \tan(x+C)$ related to $y = \tan(x)$ shifted horizontally by $(-C)$

1.4 Separable DE's: Important applications, as well as a lot of the examples we study in slope field discussions of section 1.3 are separable DE's. So let's discuss precisely what they are, and why the separation of variables algorithm works.

Definition: A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = \underline{f(x)\phi(y)} \,.$$

It's more convenient to rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad \text{(as long as } \phi(y) \neq 0\text{)} \,.$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$g(y)\frac{dy}{dx} = f(x) \,.$$

Solution (math justified): The left side of the modified differential equation is short for $g(y(x))\dfrac{dy}{dx}$. And if $G(y)$ is any antiderivative of $g(y)$, then we can rewrite this as

$$G'(y(x))y'(x)$$

which by the chain rule (read backwards) is nothing more than

$$\frac{d}{dx}G(y(x)).$$

And the solutions to

$$\frac{d}{dx}G(y(x)) = f(x)$$

are

$$G(y(x)) = \int f(x)\,dx = F(x) + C.$$

where $F(x)$ is any antiderivative of $f(x)$. Thus solutions $y(x)$ to the original differential equation satisify

$$G(y) = F(x) + C \,.$$

This expresses solutions $y(x)$ implicitly as functions of x. You may be able to use algebra to solve this equation explicitly for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

Solution (differential magic): Treat $\dfrac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)dy = f(x)dx \,.$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx \text{, i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C \,. \quad \text{Agrees!}$$

This is the same differential magic that you used for the "method of substitution" in antidifferentiation, which was essentially the "chain rule in reverse" for integration techniques.

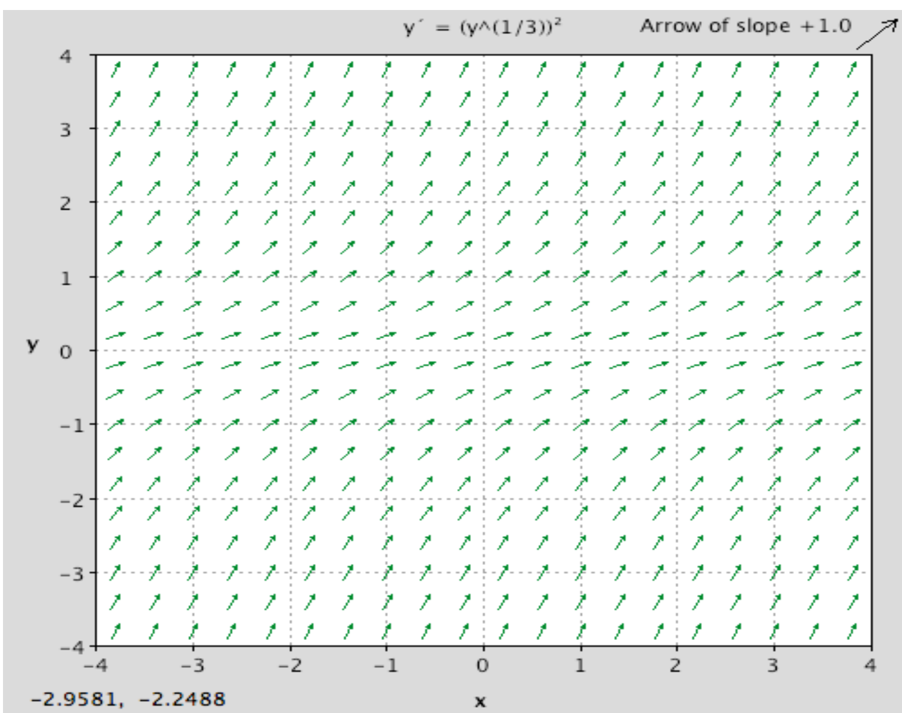
Exercise 2a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$

$$y(0) = 0$$

2b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

2c) Sketch some of these singular solutions onto the slope field below.



Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)

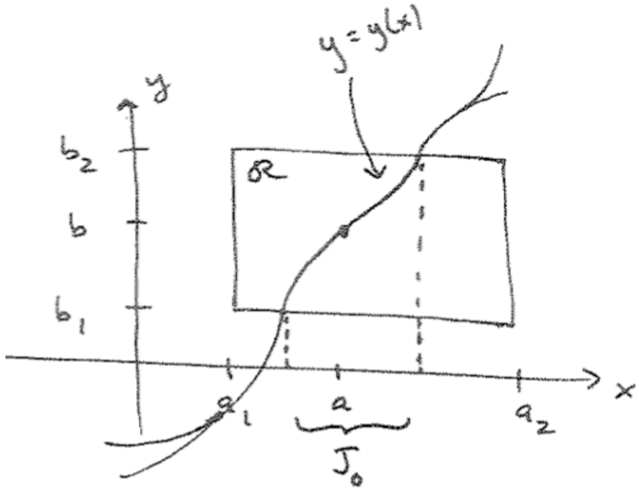
Existence - uniqueness theorem for the initial value problem

Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the $x-y$ plane.
- Existence: If $f(x, y)$ is continuous in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.
- Uniqueness: If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



Exercise 3: Discuss how the existence-uniqueness theorem is consistent with our work in Wednesday's Exercises 1-2, and in today's Exercises 1-2 where we were able to find explicit solution formulas because the differential equations were actually separable.