Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = k\,P(M - P)$$

$P > M \Rightarrow P'(t) = \underset{P}{(+)}\underset{M-P}{(-)} = (-)$
$\Rightarrow P(t)$ is decreasing

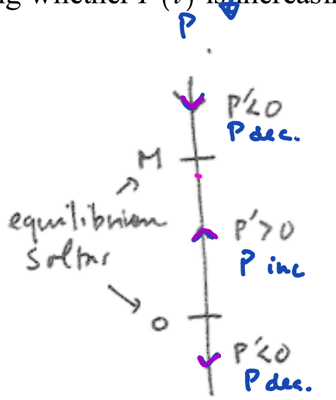
$0 < P < M \Rightarrow P'(t) = (+)(+) = (+)$
$\Rightarrow P(t)$ is increasing

$P < 0 \Rightarrow P'(t) = (-)(+) = (-)$
$\Rightarrow P(t)$ is decreasing.

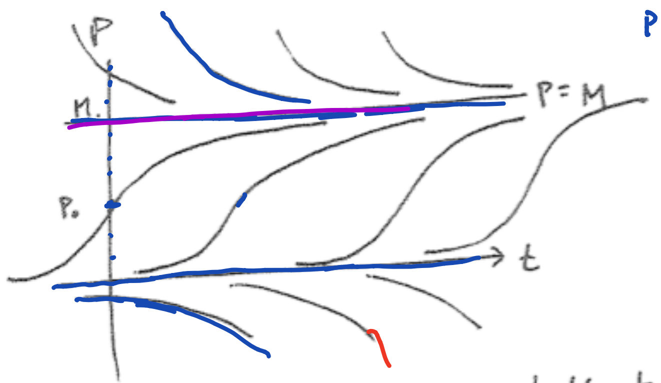
a) There are two constant ("equilibrium") solutions. What are they?

$P = 0$ const. soln.
$P = M$ const. soln.

b) Evaluate the sign and magnitude of the slope function $f(P, t) = k\,P(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualititative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram , is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.

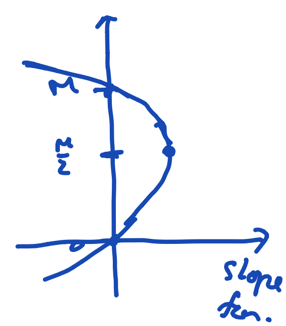


phase portrait



slope field, as represented by typical solution graphs

P > M



$kP(M-P)$

Wed start here.

if any of solution graphs touched at common point (t_1, P_1) then IVP there would not have a unique solution.

$kMP - kP^2$

slope fcn $f(t,P) = kP(M-P)$ contin (t,P)

$\frac{\partial f}{\partial P} = kM - 2kP$ cont in (t,P)

c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M . Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$?

$\boxed{\text{solution are unique}}$

M carrying capacity

example

$$\begin{cases} x'(t) = 3x(x-5) \\ x(0) = 2 \end{cases}$$

separate:

$$\int \frac{dx}{x(x-5)} = \int 3\,dt \qquad (x \neq 0, 5)$$

$$5\int \frac{1}{5}\left(\frac{1}{x-5} - \frac{1}{x}\right)dx = 5\int 3\,dt$$

$$\int \frac{1}{x-5} - \frac{1}{x}\,dx = \int 15\,dt$$

$$\ln|x-5| - \ln|x| = 15t + C_1$$

$$e^{\ln\left|\frac{x-5}{x}\right|} = e^{15t + C_1}$$

$$\left|\frac{x-5}{x}\right| = e^{15t}e^{C_1}$$

$$\frac{x-5}{x} = Ce^{15t} \qquad C = \pm e^{C_1}$$

@ $t=0$: $-\frac{3}{2} = Ce^{0} = C.$

$x(0)=2$

$$x \cdot \frac{x-5}{x} = -\frac{3}{2}e^{15t} \cdot x$$

$$x - 5 = x\left(-\frac{3}{2}e^{15t}\right)$$

$$x + \frac{3}{2}e^{15t}x = 5$$

$$x\left(1 + \frac{3}{2}e^{15t}\right) = 5$$

$$x = \frac{5}{1 + \frac{3}{2}e^{15t}} \cdot \frac{2}{2} = \frac{10}{2 + 3e^{15t}}$$

partial fractions

$$\frac{1}{x(x-5)} = \frac{A}{x} + \frac{B}{x-5} \qquad \frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

shortcut:

$$\frac{1}{5}\left(\frac{1}{x-5} - \frac{1}{x}\right)$$

$$\frac{1}{5}\,\frac{\cancel{x} - (\cancel{x}-5)}{x(x-5)}$$

$$\frac{1}{\cancel{5}}\,\frac{\cancel{5}}{x(x-5)}$$

$$= \frac{1}{5}\left(\frac{1}{x-5} - \frac{1}{x}\right)$$

shortcut

long way:

mult both sides by $x(x-5)$

$$1 = A(x-5) + Bx$$

@ $x=0$: $1 = -5A \Rightarrow A = -\frac{1}{5}$

@ $x=5$: $1 = 5B \Rightarrow B = \frac{1}{5}$

$$\frac{1}{x(x-5)} = -\frac{1}{5}\,\frac{1}{x} + \frac{1}{5}\,\frac{1}{x-5}$$

Exercise 2: Solve the logistic DE IVP

$$P' = k\,P(M - P)$$
$$P(0) = P_0$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0} .$$

Solution (we will work this out step by step in class):

$$\frac{dP}{P(P - M)} = -k\,dt$$

By partial fractions,

$$\frac{1}{P(P - M)} = \frac{1}{M}\left(\frac{1}{P - M} - \frac{1}{P}\right).$$

Use this expansion and multiply both sides of the separated DE by M to obtain

$$\left(\frac{1}{P - M} - \frac{1}{P}\right)dP = -kM\,dt .$$

Integrate:

$$\ln|P - M| - \ln|P| = -Mkt + C_1$$
$$\ln\left|\frac{P - M}{P}\right| = -Mkt + C_1$$

exponentiate:

$$\left|\frac{P - M}{P}\right| = C_2\,e^{-Mkt}$$

Since the left-side is continuous

$$\frac{P - M}{P} = C\,e^{-Mkt} \quad (C = C_2 \text{ or } C = -C_2)$$

(At $t = 0$ we see that

$$\frac{P_0 - M}{P_0} = C .)$$

Now, solve for $P(t)$ by multiplying both sides of of the second to last equation by $P(t)$:

$$P - M = Ce^{-Mkt}P$$

Collect $P(t)$ terms on left, and add M to both sides:

$$P - Ce^{-Mkt}P = M$$
$$P\left(1 - Ce^{-Mkt}\right) = M$$
$$P = \frac{M}{1 - Ce^{-Mkt}} .$$

Plug in C and simplify:

$$P = \frac{M}{1 - \left(\frac{P_0 - M}{P_0}\right)e^{-Mkt}} = \frac{MP_0}{P_0 - (P_0 - M)e^{-Mkt}}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}}\,.$$

$$\lim_{t \to \infty} e^{-Mkt} = 0$$

Finally, because $\lim_{t \to \infty} e^{-Mkt} = 0$, we see that

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

solution formula verifies the phase diagram & slope field pictures

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0, for $t > 0$.)

Question: You have a couple of homework problems where you are asked to find solutions $x(t)$ to differential equations of the form

$$x'(t) = a(x - b)\cdot(x - c).$$

How would you proceed?

$$\frac{dx}{dt} = a(x-b)(x-c)$$

$$\int \frac{dx}{(x-b)(x-c)} = \int a\,dt$$

$$\frac{1}{(x-b)(x-c)} = \frac{A}{x-b} + \frac{B}{x-c}$$

$$\int \frac{A}{x-b} + \frac{B}{x-c}\,dx = \int a\,dt$$

$$A\ln|x-b| + B\ln|x-c| = at + C$$

$$e \qquad\qquad e$$

$$|x-b|^A\,|x-c|^B = e^{at}e^{C}$$

...

Application!

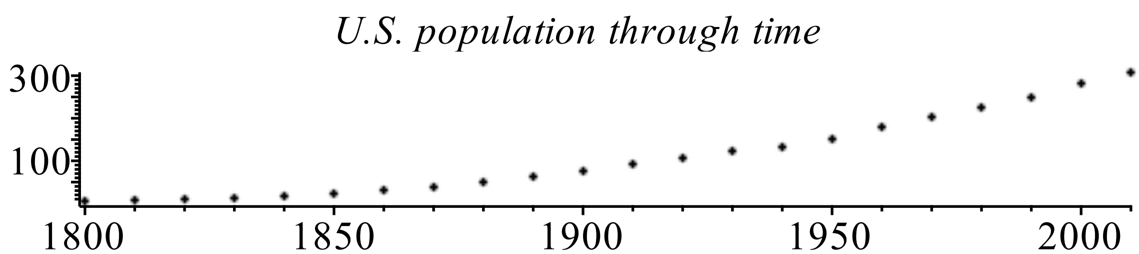
The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
```

Verhulst

```
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
     [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
     [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
     [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
     [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
     [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
     # I used shift-enter to enter more than one line of information
     # before executing the command.
```

```
> with(plots) : # plotting library of commands
  pointplot(pops, title = `U.S. population through time`);
```




```
>
```

Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let t=0 correspond to 1800.

$t=0$

$P_0 = 5.3$

Exponential Model: For the exponential growth model $P(t) = P_0 e^{rt}$ we use the 1800 and 1900 data to get values for P_0 and r :

$P(100) = 76.2$ million.

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
```

$$P0 := 5.308$$

$$0.026643 \tag{2}$$

```
> P1 := t→5.308· exp(.02664· t);#exponential model -eqtn (9) page 83
```

$$P1 := t \to 5.308\, e^{0.02664\, t} \tag{3}$$

```
>
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

```
> P2 := t→M·P0/(P0 + (M−P0)·exp(−M·k·t)); # logistic solution we worked out
```

$$P2 := t \to \frac{M\,P0}{P0 + (M - P0)\,\mathrm{e}^{-M k t}} \tag{4}$$

```
> solve({P2(50) = 23.192, P2(100) = 76.212}, {M, k});
```

$$\{M = 188.12,\ k = 0.00016772\} \tag{5}$$

```
> M := 188.12;
  k := .16772e-3;
  P2(t); #should be our logistic model function,
     #equation (11) page 84.
```

$$M := 188.12$$

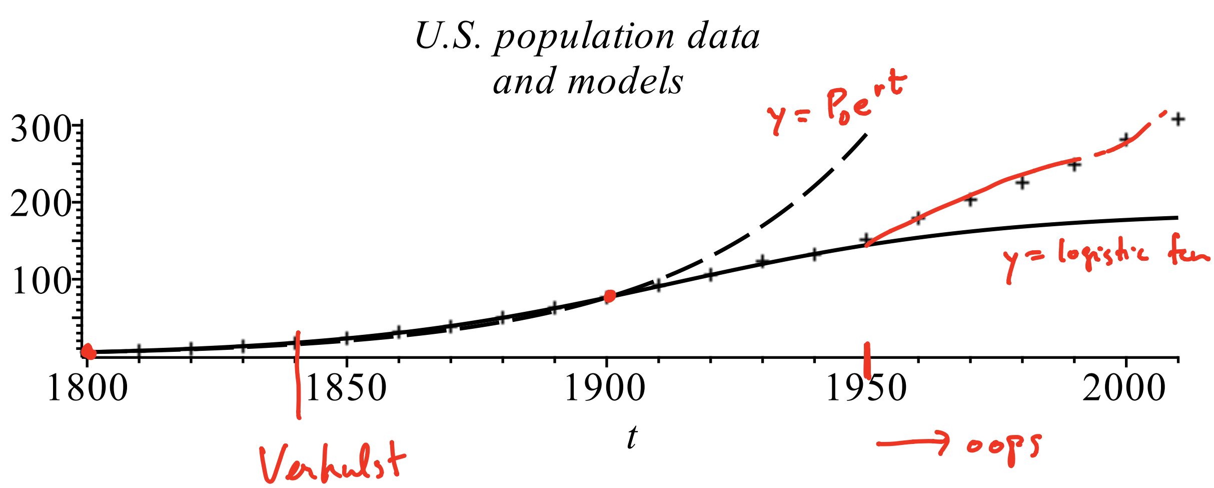
$$k := 0.00016772$$

$$\frac{998.54}{5.308 + 182.81\,\mathrm{e}^{-0.031551\,t}} \tag{6}$$

```
>
```

Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

```
> plot1 := plot(P1(t−1800), t = 1800..1950, color = black, linestyle = 3) :
   #this linestyle gives dashes for the exponential curve
  plot2 := plot(P2(t−1800), t = 1800..2010, color = black) :
  plot3 := pointplot(pops, symbol = cross) :
  display({plot1, plot2, plot3}, title = `U.S. population data
  and models`);
```




```
>
```

baby boom.

better resources — fertility rates went up.

longer lives — mortality rates went down in mid 1900's.

- HW hand-in & pick-up up front
- quiz at end of class
- finish §2.1
- start §2.2 pages 1–3 today's notes

Math 2250-004
Wednesday Jan 25

2.2: Autonomous Differential Equations.
Recall, that if we solve for the derivative, a general first order DE for $x = x(t)$ is written as

$$x' = f(t, x) \; ,$$

slope fcn depends on x & t

which is shorthand for $x'(t) = f(t, x(t))$.
Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation *autonomous*:

$$x' = f(x) \; .$$

(rate of change) $P'(t) = kP(M-P)$
slope only depends on x

Example: The logistic DE, $P' = k\,P(M - P)$ is an autonomous differential equation for $P(t)$, for example.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called *equilibrium solutions*. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

$x(t) \equiv c$ solution
$x'(t) = 0 = f(x) = f(c)$

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of
1a) $x'(t) = 3x - x^2$ $= x(3-x)$

equil solns are $x(t) \equiv 0$
$x(t) \equiv 3$

1b) $x'(t) = x^3 + 2x^2 + x$ $= x(x^2+2x+1)$
$= x(x+1)^2$

equil solns $x \equiv 0$
$x \equiv -1$

1c) $x'(t) = \sin(x)$.