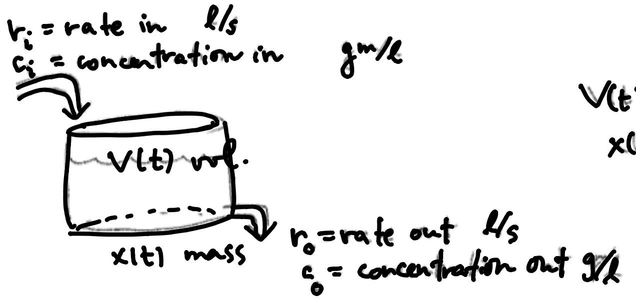
An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i ("rate in") (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o ("rate out"), and with concentration c_0. We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t, given $V(0) = V_0$, $x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l}.$$

← average concentration in tank.

See the diagram below.



V(t) = volume in tank at time t (l)
x(t) = amount of solute in tank (gm)
$c(t) = \frac{x(t)}{V(t)}$ g/l (average) concentration in tank

Exercise 4: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0$$

§1.2 $r_i(t)$, $r_o(t)$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

$V(t) = \int r_i - r_o \, dt$, then find C or, use definite integral

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

consider a small time increment Δt, estimate Δx

Δx = what comes in − what goes out

$\Delta x \approx r_i \, \Delta t \, c_i - r_o \, \Delta t \, c_o$

(r_i: vol/time; Δt: time; c_i: mass/vol; $r_i \Delta t$: ΔV_{in}; $r_o \Delta t$: ΔV_o)

e.g. $r_i = 10$ l/min
$\Delta t = .1$ min
$c_i = 5$ g/l
$\Delta x_{in} = (10)(.1) \cdot 5 = 5$ g
i.e. g/l

$\div \Delta t$: $\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o$

$\lim_{\Delta t \to 0}$: $x'(t) = r_i c_i - r_o c_o$

$x'(t) = r_i c_i - r_o \frac{x(t)}{V(t)}$

$V'(t) = r_i - r_o$

$\int_0^t V'(\tau)\, d\tau = \int_0^t r_i(\tau) - r_o(\tau)\, d\tau$

$V(\tau)\Big]_0^t$

$V(t) - V(0) = \int_0^t r_i(\tau) - r_o(\tau)\, d\tau$

$V(t) = V_0 + \int_0^t r_i(\tau) - r_o(\tau)\, d\tau$

if const $= r_i = r_o \Rightarrow V$ is constant

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$\begin{aligned} x' + a\,x &= b \\ x(0) &= x_0 \end{aligned}$$

where a, b are constants. This differential equation is separable and linear, and it is recommended that you become good at solving it. Notice that it includes the exponential growth/decay and Newton's law of cooling DE's as special cases.

Math 2250-004
Week 3 notes, Jan 23-27

• input-output modeling in Friday notes
• Exercise 1 today's notes 1st
• then application in today's notes

Mon Jan 23
1.5: linear DEs, and applications.

Recall from Friday notes (review/cover if necessary) that input-out models often lead to the IVP

$x = x(t)$:

$$x' + a\,x = b$$
$$x(0) = x_0$$

$x'(t) + P(t)\,x(t) = Q(t)$

where a, b are (positive) constants.

Exercise 1: The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. We might check our answer with Wolfram alpha. Maple check is below.

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}.$$

(1) $P(t) = a$

I.F. $e^{\int P(t)dt} = e^{\int a\,dt} = e^{at}$

$e^{at}\left[x' + ax\right] = e^{at}b$ ← LHS $e^{at}x' + e^{at}ax$ — same.

(2) $\frac{d}{dt}\left(e^{at}x(t)\right) = e^{at}b.$ ($f = e^{at}$, $g = x(t)$)

← LHS via product rule $(fg)' = f'g + fg' = ae^{at}x + e^{at}x'$

(3) integrate: $e^{at}x = \int b e^{at}\,dt$

$e^{at}x = b\frac{e^{at}}{a} + C$

(4) ÷ I.F. $x = \frac{b}{a} + Ce^{-at}$

(5) IVP: $x(0) = x_0$: $x_0 = \frac{b}{a} + C \Rightarrow C = x_0 - \frac{b}{a}.$

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}$$

```
> with(DEtools) :
  dsolve({x'(t) + a·x(t) = b, x(0) = x0});
```

$$x(t) = \frac{b}{a} + e^{-a\,t}\left(x0 - \frac{b}{a}\right) \qquad (1)$$

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>
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Exercise 2: Use the result above to solve a pollution problem IVP and answer the following question (p. 55-56 text): Lake Huron has a pretty constant concentration for a certain pollutant. Due to an industrial accident, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of $480\ km^3$, and water flows into and out of Lake Erie at a rate of $350\ km^3$ per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the background concentration from Lake Huron?

"c" mass/vol. 5c 2c.

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \frac{km^3}{y}$ $= r_i = r_o$

b) Solve the IVP, and then answer the question.

$x'(t) = r_i c_i - r_o c_o$ (= av concentration in lake (at time t) $\frac{x}{V}$)

$x'(t) = rc - \frac{r}{V}x$

$$\begin{cases} x'(t) + \frac{r}{V}x(t) = rc \\ x(0) = 5cV \end{cases}$$

initial concentration mass/vol.

$$\begin{cases} x'(t) + ax = b \\ x(0) = x_0 \end{cases}$$

soln

$x(t) = \frac{b}{a} + (x_0 - \frac{b}{a})e^{-at}$

$a = \frac{r}{V}$ $\quad \frac{b}{a} = \frac{rc}{r/V} = cV$

$b = rc$

$x(t) = cV + (5cV - cV)e^{-\frac{r}{V}t}$

$x(t) = cV + 4cVe^{-\frac{r}{V}t}$

set $x(t) = 2cV$ $\quad 2cV = cV + 4cVe^{-\frac{r}{V}t}$

$\div cV$ $\quad 2 = 1 + 4e^{-\frac{r}{V}t} \Longrightarrow 1 = 4e^{-\frac{350}{480}t}$

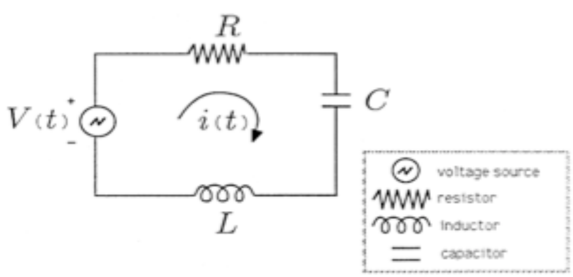
$.25 = e^{-350/450\, t}$

$\ln(.25) = -\frac{350}{450} t$

$t \cong 1.9$ years

EP 3.7 This is a supplementary section. I've posted a .pdf on our homework page.

Often the same DE can arise in completely different-looking situations. For example, first order linear DE's also arise (as special cases of second order linear DE's) in simple *RLC* circuit modeling.



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e $Q'(t) = I(t)$. $Q'' = I'$

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

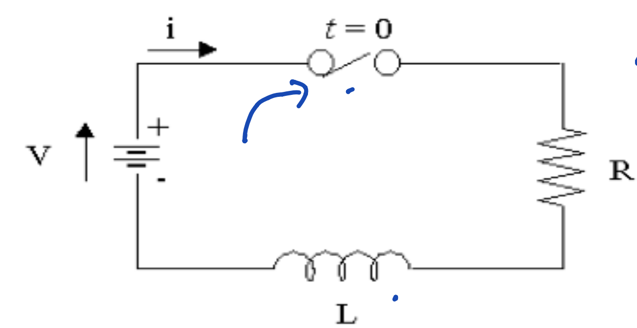
For $Q(t)$: $$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t)$$

$\frac{d}{dt}$ & Kirchoff

For $I(t)$: $$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t)$$

if no inductor, or if no capacitor, then Kirchoff's Law yields 1st order linear DE's, as below:

Exercise 3: Consider the $R-L$ circuit below, in which a switch is thrown at time $t = 0$. Assume the voltage V is constant, and $I(0) = 0$. Find $I(t)$. Interpret your results.



http://www.intmath.com/differential-equations/5-rl-circuits.php

$L\,I'(t) + R\,I(t) = V$

$\begin{cases} I'(t) + \frac{R}{L} I(t) = \frac{V}{L} \\ I(0) = 0 \end{cases}$

$\begin{cases} x'(t) + a\,x(t) = b \\ x(0) = x_0 \end{cases}$ a, b const.

$x(t) = \frac{b}{a} + (x_0 - \frac{b}{a}) e^{-at}$

↑ const soln

transpose: $\frac{b}{a} = \frac{V/L}{R/L} = \frac{V}{R}$

$\Rightarrow I(t) = \frac{V}{R} + (0 - \frac{V}{R}) e^{-\frac{R}{L}t}$

$I(t) = \frac{V}{R}(1 - e^{-\frac{R}{L}t})$

learn in physics

decays quickly

• finish §1.5, RLC circuits (Monday)
EP 3.7

• begin 2.1-2.2 (today's notes)
• Quiz: 1.4-1.5

Tues Jan 24
2.1 Improved population models

Let $P(t)$ be a population at time t. Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{people}{year}$);

$$\beta(t) := \frac{B(t)}{P(t)}, \text{ fertility rate } \left(\frac{people}{year} \text{ per } person\right)$$ $B(t) = \beta(t)P(t)$

$\mathrm{D}(t)$, death rate (e.g. $\frac{people}{year}$);

$$\delta(t) := \frac{\mathrm{D}(t)}{P(t)}, \text{ mortality rate } \left(\frac{people}{year} \text{ per } person\right)$$ $D(t) = \delta(t)P(t)$

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - \mathrm{D}(t)$$
$$P'(t) = (\beta(t) - \delta(t))P(t).$$

Model 1: constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \ge 0$, $\delta(t) \equiv \delta_0 \ge 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = kP. \quad \Rightarrow P(t) = P_0 e^{kt}.$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

Model 2: population fertility and mortality rates only depend on population P, but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$
$$\delta = \delta_0 + \delta_1 P$$

} not allowing for changes that just have to do with time.

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = \left((\beta_0 + \beta_1 P) - (\delta_0 + \delta_1 P)\right)P$$
$$P' = \left((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P\right)P.$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect $\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a$ < 0, $\beta_0 - \delta_0 = b > 0$ one obtains the logistic differential equation:

$$P' = (b - aP)P$$
$$P' = bP - aP^2, \text{ or equivalently}$$
$$P' = aP\left(\frac{b}{a} - P\right) = kP(M - P).$$

$k = a > 0$, $M = \frac{b}{a} > 0$. (One can consider other cases as well.)

$P'(t) = kP(M - P)$

Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = k\,P(M - P)$$

$P > M \Rightarrow P'(t) = (+)(-) = (-) \Rightarrow P(t)$ is decreasing

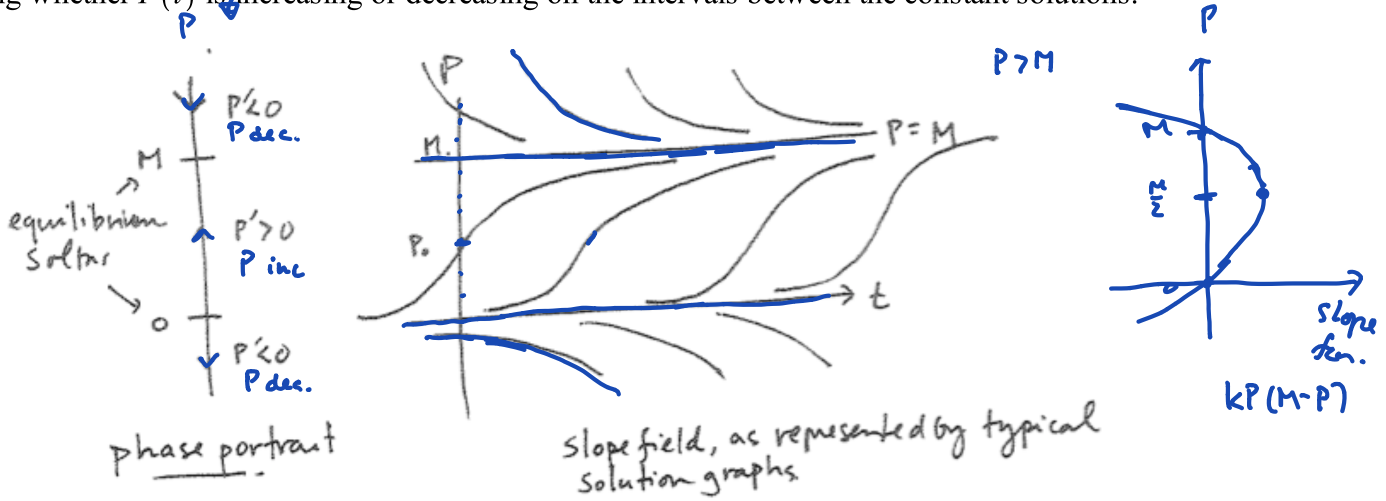
$0 < P < M \Rightarrow P'(t) = (+)(+) = (+) \Rightarrow P(t)$ is increasing

$P < 0 \Rightarrow P'(t) = (-)(+) = (-) \Rightarrow P(t)$ is decreasing.

a) There are two constant ("equilibrium") solutions. What are they?

$P = 0$ const. soltn.
$P = M$ const. soltn.

b) Evaluate the sign and magnitude of the slope function $f(P, t) = k\,P(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualititative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram, is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.



if any of solution graphs touched at common point (t_1, P_1) then IVP there would not have a unique solution.

$kMP - kP^2$

slope fn $f(t,P) = kP(M-P)$ cont in (t,P)

$\frac{\partial f}{\partial P} = kM - 2kP$ cont in (t,P)

solution exists & unique

c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M. Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \to \infty} P(t)$?

M carrying capacity

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}.$$

Finally, because $\lim_{t \to \infty} e^{-Mkt} = 0$, we see that

$$\lim_{t \to \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0, for $t > 0$.)

Question: You have a couple of homework problems where you are asked to find solutions $x(t)$ to differential equations of the form

$$x'(t) = a(x - b) \cdot (x - c).$$

How would you proceed?

$$\frac{dx}{dt} = a(x-b)(x-c)$$

$$\int \frac{dx}{(x-b)(x-c)} = \int a\, dt$$

$$\frac{1}{(x-b)(x-c)} = \frac{A}{x-b} + \frac{B}{x-c}$$

$$\int \frac{A}{x-b} + \frac{B}{x-c}\, dx = \int a\, dt$$

$$A \ln|x-b| + B \ln|x-c| = at + C$$

$$|x-b|^A \, |x-c|^B = e^{at} e^C$$

...