An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o, and with concentration c_0. We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t, given $V(0) = V_0$, $x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

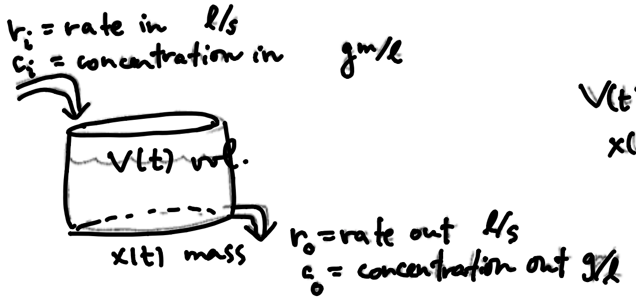
"rate in"

"rate out"

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l}.$$

average concentration in tank.

See the diagram below.



$V(t)$ = volume in tank at time t (l)

$x(t)$ = amount of solute in tank (gm)

$c(t) = \frac{x(t)}{V(t)}$ $\frac{g}{l}$ (average) concentration in tank

Exercise 4: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0$$

§1.2

$r_i(t)$
$r_o(t)$

$$\text{so } \boxed{V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau}$$

$V(t) = \int r_i - r_o \, dt$, then find C

or, use definite integral

$V'(t) = r_i - r_o$

$\int_0^t V'(\tau)\,d\tau = \int_0^t r_i(\tau) - r_o(\tau)\,d\tau$

$V(\tau)\Big]_0^t$

$V(t) - V(0) = \int_0^t r_i(\tau) - r_o(\tau)\,d\tau$

$V(t) = V_o + \int_0^t r_i(\tau) - r_o(\tau)\,d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$\boxed{x'(t) + \frac{r_o}{V} x(t) = r_i c_i}$$

consider a small time increment Δt, estimate Δx

Δx = what comes in − what goes out

$$\boxed{\Delta x \approx \underbrace{\underbrace{r_i}_{\text{vol/time}}\ \underbrace{\Delta t}_{\text{time}}}_{\Delta V_{in}}\ \underbrace{c_i}_{\text{mass/vol.}} - \underbrace{r_o\, \Delta t}_{\Delta V_o}\ c_o}$$

e.g. $r_i = 10$ l/min
$\Delta t = .1$ min
$c_i = 5$ g/l
$\Delta x_{in} = (10)(.1)\cdot 5 = 5$ g
i.e. g/l

$\div \Delta t$: $\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o$

$\lim_{\Delta t \to 0}$ $x'(t) = r_i c_i - r_o c_o$

$$\boxed{x'(t) = r_i c_i - r_o \frac{x(t)}{V(t)}}$$

if const $= r_i = r_o \Rightarrow V$ is constant

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + \underbrace{a}_{\frac{r_o}{V}}\,x = \underbrace{b}_{r_i c_i}$$
$$x(0) = x_0$$

where a, b are constants. This differential equation is separable and linear, and it is recommended that you become good at solving it. Notice that it includes the exponential growth/decay and Newton's law of cooling DE's as special cases.

Math 2250-004
Week 3 notes, Jan 23-27

• input-output modeling in Friday notes
• Exercise 1 today's notes 1st
• then application in today's notes

Mon Jan 23
1.5: linear DEs, and applications.

Recall from Friday notes (review/cover if necessary) that input-out models often lead to the IVP

$x = x(t)$:

$$\boxed{\begin{aligned} x' + \underline{a}\, x &= b \\ x(0) &= x_0 \end{aligned}}$$

$x'(t) + \underline{P(t)}\, x(t) = Q(t)$

where $\underline{a}$, b are (positive) <u>constants</u>.

<u>Exercise 1:</u> The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. We might check our answer with Wolfram alpha. Maple check is below.

$$x(t) = \frac{b}{a} + \left(x_o - \frac{b}{a}\right)e^{-a\,t}.$$

① $P(t) = a$

I.F. $e^{\int P(t)dt} = e^{\int a\,dt} = e^{at}$

$e^{at}\left[x' + ax\right] = e^{at}b$ ← LHS $e^{at}x' + e^{at}ax$

② $\frac{d}{dt}\left(\underbrace{e^{at}}_{f}\underbrace{x(t)}_{g}\right) = e^{at}b.$ ← LHS via product rule

$(fg)' = f'g + fg'$

$= ae^{at}x + e^{at}x'$

same.

③ integrate: $e^{at}x = \int be^{at}\,dt$

$e^{at}x = b\frac{e^{at}}{a} + C$

④ ÷ I.F. $x = \frac{b}{a} + Ce^{-at}$

⑤ IVP: $x(0) = x_0$: $x_0 = \frac{b}{a} + C \Rightarrow C = x_0 - \frac{b}{a}.$

$$\boxed{x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}}$$

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> with(DEtools) :
  dsolve({x'(t) + a·x(t) = b, x(0) = x0});
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$$\boxed{x(t) = \frac{b}{a} + \mathrm{e}^{-a\,t}\left(x0 - \frac{b}{a}\right)} \qquad \textbf{(1)}$$

>

"c"   mass/vol.

Exercise 2: Use the result above to solve a pollution problem IVP and answer the following question (p. 55-56 text): Lake Huron has a pretty constant concentration for a certain pollutant. Due to an industrial accident, Lake Erie has suddenly obtained a concentration five times (5c) as large. Lake Erie has a volume of $480\ km^3$, and water flows into and out of Lake Erie at a rate of $350\ km^3$ per year. Essentially all of the in-flow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the background concentration from Lake Huron? (2c.)

http://www.enchantedlearning.com/usa/statesbw/greatlakesbw.GIF

a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

$V = 480\ km^3$

$r = 350\ \frac{km^3}{y} = r_i = r_o$

b) Solve the IVP, and then answer the question.

c_o = av concentration in lake (at time t) $= \frac{x}{V}$

$$x'(t) = r_i c_i - r_o c_o$$
$$x'(t) = rc - \frac{r}{V}x$$

$$\begin{cases} x'(t) + \frac{r}{V}x(t) = rc \\ x(0) = 5cV \end{cases}$$

(5c: initial concentration; mass = concentration · vol.)

$$\begin{cases} x'(t) + ax = b \\ x(0) = x_0 \end{cases}$$

soln

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}$$

$a = \frac{r}{V}$ $\quad \frac{b}{a} = \frac{rc}{r/V} = cV$

$b = rc$

$$x(t) = cV + (5cV - cV)e^{-\frac{r}{V}t}$$
$$x(t) = cV + 4cVe^{-\frac{r}{V}t}$$

set $x(t) = 2cV$ $\quad 2cV = cV + 4cVe^{-\frac{r}{V}t}$

$\div cV$ $\quad 2 = 1 + 4e^{-\frac{r}{V}t} \Longrightarrow 1 = 4e^{-\frac{350}{480}t}$

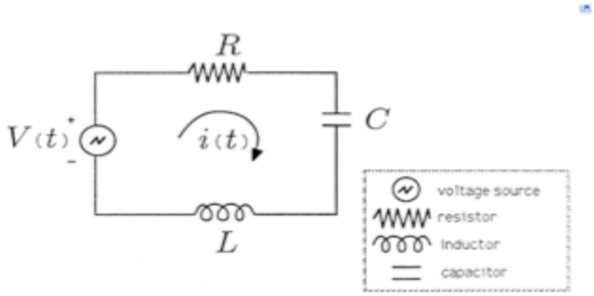
$.25 = e^{-350/450\, t}$

$\ln(.25) = -\frac{350}{450}\, t$

$t \cong 1.9$ years

EP 3.7 This is a supplementary section. I've posted a .pdf on our homework page.

Often the same DE can arise in completely different-looking situations. For example, first order linear DE's also arise (as special cases of second order linear DE's) in simple *RLC* circuit modeling.



circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C}Q(t)$	C Farads (F)

http://cnx.org/content/m21475/latest/pic012.png

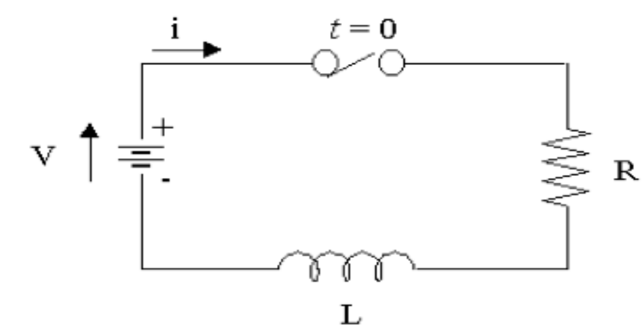
Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e $Q'(t) = I(t)$.
Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

For $Q(t)$: $$L\,Q''(t) + R\,Q'(t) + \frac{1}{C}Q(t) = V(t)$$

For $I(t)$: $$L\,I''(t) + R\,I'(t) + \frac{1}{C}I(t) = V'(t)$$

if no inductor, or if no capacitor, then Kirchoff's Law yields 1^{st} order linear DE's, as below:

Exercise 3: Consider the $R - L$ circuit below, in which a switch is thrown at time $t = 0$. Assume the voltage V is constant, and $I(0) = 0$. Find $I(t)$. Interpret your results.



http://www.intmath.com/differential-equations/5-rl-circuits.php