

Exercise 3: Find all solutions to the linear (and also separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2 + 1} = \frac{6x}{x^2 + 1} - \frac{3x}{x^2 + 1} y$$

Hint: as you can verify below, the general solution is $y(x) = 2 + C(x^2 + 1)^{-\frac{3}{2}}$.

> with (DEtools) :

$$\text{dsolve}\left(y'(x) = \frac{(6 \cdot x - 3 \cdot x \cdot y(x))}{x^2 + 1}, y(x)\right); \text{ \#Maple check}$$

$$y(x) = 2 + \frac{C1}{(x^2 + 1)^{3/2}}$$

(8)

$$y' + \frac{3x}{x^2 + 1} y = \frac{6x}{x^2 + 1}$$

$P(x)$

⑤ ÷ I.F.

$$y = 2 + C(x^2 + 1)^{-3/2}$$

$$\textcircled{1} \int P(x) dx = \int \frac{3x}{x^2 + 1} dx = \int \frac{3}{2} \frac{du}{u} = \frac{3}{2} \ln|u| + e$$

$$u = x^2 + 1$$

$$du = 2x dx$$

$$\frac{3}{2} du = 3x dx$$

$$= \frac{3}{2} \ln(x^2 + 1)$$

$$\textcircled{2} \text{ I.F. } e^{\int P(x) dx} = e^{\frac{3}{2} \ln(x^2 + 1)} = \left[e^{\ln(x^2 + 1)} \right]^{\frac{3}{2}}$$

$$= (x^2 + 1)^{3/2}$$

$$\textcircled{3} (x^2 + 1)^{3/2} \left[y' + \frac{3x}{x^2 + 1} y \right] = (x^2 + 1)^{3/2} \cdot \frac{6x}{x^2 + 1}$$

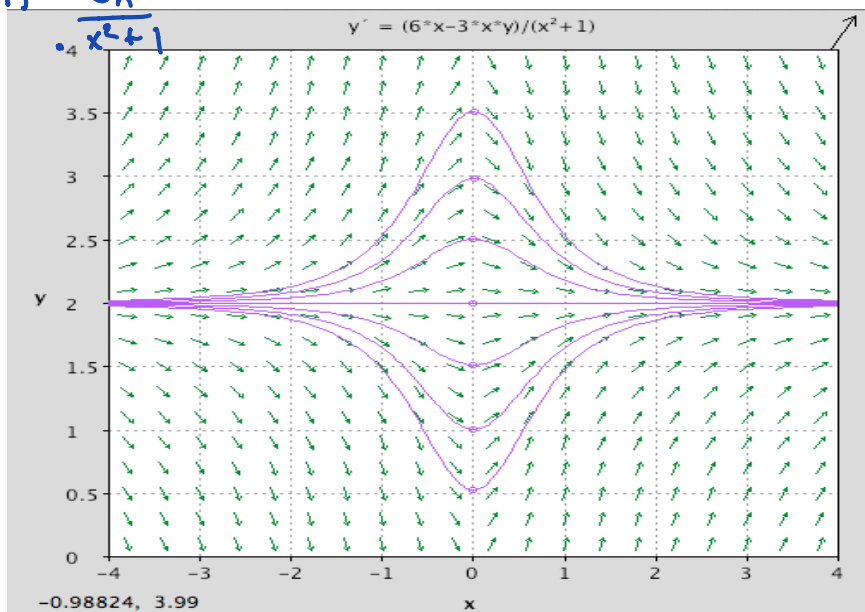
$$\textcircled{4} \frac{d}{dx} \left[(x^2 + 1)^{3/2} y \right] = 6x (x^2 + 1)^{1/2}$$

$$f' = \frac{3}{2} (x^2 + 1)^{1/2} \cdot 2x$$

$$(x^2 + 1)^{3/2} y = \int 6x (x^2 + 1)^{1/2} dx$$

$$(x^2 + 1)^{3/2} y = 2(x^2 + 1)^{3/2} + C$$

$$\frac{d}{dx} 2(x^2 + 1)^{3/2} = \frac{3}{2} (x^2 + 1)^{1/2} \cdot 2x$$



$$\frac{dy}{dx} = \frac{3x}{x^2+1} (2-y)$$

$$* \frac{dy}{y-2} = -\frac{3x}{x^2+1} dx \quad \left(y \neq 2 \rightarrow y(x) \equiv 2 \text{ is soln} \right)$$

$$y' = 0, \text{ RHS} = 0$$

I'll fill in the rest!

filled in: integrate *

$$\int \frac{dy}{y-2} = \int \frac{-3x}{x^2+1} dx \quad \leftarrow \begin{array}{l} u = x^2+1 \\ du = 2x dx \end{array}$$

$$\downarrow$$

$$\ln|y-2| = -\frac{3}{2} \ln(x^2+1) + C$$

$$\left(u = y-2, du = dy \right) \quad \begin{array}{l} -\frac{3}{2} du = -3x dx \\ \int \frac{-\frac{3}{2} du}{u} = -\frac{3}{2} \ln|u| = -\frac{3}{2} \ln(x^2+1) \end{array}$$

exponentiate:

$$|y-2| = e^{-\frac{3}{2} \ln(x^2+1)} e^C$$

$$y-2 = C (x^2+1)^{-3/2}$$

$$\boxed{y = 2 + C (x^2+1)^{-3/2}}$$

$$C = \pm e^C$$

note $C=0$ yields the singular soln $y=2$

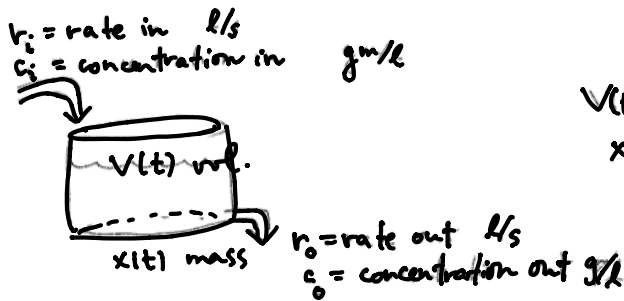
An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o , and with concentration c_o . We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t , given $V(0) = V_0$, $x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \frac{gm}{l}$$

↑ average concentration in tank.

See the diagram below.



$V(t)$ = volume in tank at time t (l)
 $x(t)$ = amount of solute in tank (gm)
 $c(t) = \frac{x(t)}{V(t)}$ $\frac{gm}{l}$ (average) concentration in tank

Exercise 4: Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_o(\tau) d\tau$

$V'(t) = r_i - r_o$ 4.2

$V(t) = \int r_i - r_o dt$, then find C or, use definite integral

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

consider a small time increment Δt , estimate Δx

$\Delta x = \text{what comes in} - \text{what goes out}$

$$\Delta x \approx \underbrace{r_i}_{\frac{vol}{time}} \underbrace{\Delta t}_{time} \underbrace{c_i}_{\frac{mass}{vol}} - \underbrace{r_o}_{\frac{vol}{time}} \underbrace{\Delta t}_{time} \underbrace{c_o}_{\frac{mass}{vol}}$$

ΔV_{in}

e.g. $r_i = 10 \text{ l/min}$
 $\Delta t = .1 \text{ min}$
 $c_i = 5 \text{ g/l}$
 $\Delta x_{in} = \underbrace{(10)(.1)}_{1 \text{ l.}} \cdot 5 = 5 \text{ g}$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

$\div \Delta t:$ $\frac{\Delta x}{\Delta t} \approx r_i c_i - r_o c_o$

$\lim_{\Delta t \rightarrow 0}$ $x'(t) = r_i c_i - r_o c_o$
 $x'(t) = r_i c_i - r_o \frac{x(t)}{V(t)}$

$V'(t) = r_i - r_o$
 $\int_0^t V'(\tau) d\tau = \int_0^t r_i(\tau) - r_o(\tau) d\tau$
 $V(t) - V(0) = \int_0^t r_i(\tau) - r_o(\tau) d\tau$
 $V(t) = V_0 + \int_0^t r_i(\tau) - r_o(\tau) d\tau$

if $\text{const} = r_i = r_o \Rightarrow V$ is constant

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a x = b$$

$$x(0) = x_0$$

where a, b are constants. This differential equation is separable and linear, and it is recommended that you become good at solving it. Notice that it includes the exponential growth/decay and Newton's law of cooling DE's as special cases.

Mon Jan 23

1.5: linear DEs, and applications.

- input-output modeling in Friday notes
- Exercise 1 today's notes 1st
- then application in today's notes

Recall from Friday notes (review/cover if necessary) that input-out models often lead to the IVP

$$x' = x(t):$$

$$\begin{cases} x' + ax = b \\ x(0) = x_0 \end{cases}$$

$$x'(t) + \underline{P(t)} x(t) = Q(t)$$

where a, b are (positive) constants.

Exercise 1: The constant coefficient initial value problem above will recur throughout the course in various contexts, so let's solve it now. We might check our answer with Wolfram alpha. Maple check is below.

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}.$$

① $P(t) = a$
I.F. $e^{\int P(t) dt} = e^{\int a dt} = e^{at}$

$$e^{at} [x' + ax] = e^{at} b$$

② $\frac{d}{dt} (\underbrace{e^{at}}_f \underbrace{x(t)}_g) = e^{at} b.$

← LHS $e^{at} x' + e^{at} ax$
← LHS via product rule
 $(fg)' = f'g + fg'$
 $= ae^{at} x + e^{at} x'$
same.

③ integrate: $e^{at} x = \int be^{at} dt$
 $e^{at} x = b \frac{e^{at}}{a} + C$

④ \div I.F. $x = \frac{b}{a} + C e^{-at}$

⑤ IVP: $x(0) = x_0: x_0 = \frac{b}{a} + C \Rightarrow C = x_0 - \frac{b}{a}.$

$$x(t) = \frac{b}{a} + \left(x_0 - \frac{b}{a}\right)e^{-at}$$

> with(DEtools):
dsolve({x'(t) + a·x(t) = b, x(0) = x0});

$$x(t) = \frac{b}{a} + e^{-at} \left(x_0 - \frac{b}{a}\right)$$

(1)

Exercise 2: Use the result above to solve a pollution problem IVP and answer the following question (p. 55-56 text): Lake Huron has a pretty constant concentration for a certain pollutant. Due to an industrial accident, Lake Erie has suddenly obtained a concentration five times as large. Lake Erie has a volume of 480 km^3 , and water flows into and out of Lake Erie at a rate of 350 km^3 per year. Essentially all of the inflow is from Lake Huron (see below). We expect that as time goes by, the water from Lake Huron will flush out Lake Erie. Assuming that the pollutant concentration is roughly the same everywhere in Lake Erie, about how long will it be until this concentration is only twice the background concentration from Lake Huron?



a) Set up the initial value problem. Maybe use symbols c for the background concentration (in Huron),

$$V = 480 \text{ km}^3$$

$$r = 350 \frac{\text{km}^3}{\text{y}} = r_i = r_o$$

b) Solve the IVP, and then answer the question.

$$x'(t) = r_i c_i - r_o c_o$$

$$x'(t) = r c - \frac{r}{V} x$$

$$\begin{cases} x'(t) + \frac{r}{V} x(t) = r c \\ x(0) = 5cV \end{cases}$$

initial concentration mass vol.

$$\begin{cases} x'(t) + a x = b \\ x(0) = x_0 \end{cases}$$

soln

$$x(t) = \frac{b}{a} + (x_0 - \frac{b}{a}) e^{-at}$$

$$a = \frac{r}{V} \quad b = r c \quad \frac{b}{a} = \frac{r c}{r/V} = cV$$

$$x(t) = cV + (5cV - cV) e^{-\frac{r}{V} t}$$

$$x(1) = cV + 4cV e^{-\frac{r}{V} t}$$

$$\text{set } x(1) = 2cV \quad 2cV = cV + 4cV e^{-\frac{r}{V} t}$$

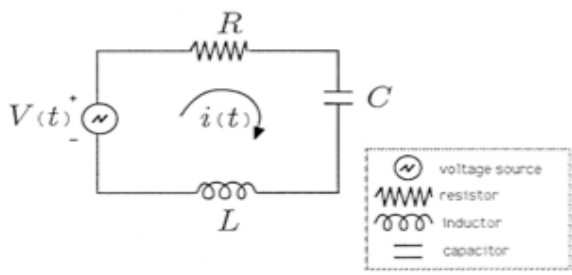
$$\div cV \quad 2 = 1 + 4e^{-\frac{r}{V} t} \implies 1 = 4e^{-\frac{350}{480} t}$$

$0.25 = e^{-350/450 t}$
 $\ln(0.25) = -\frac{350}{450} t$

$t \approx 1.9 \text{ years}$

EP 3.7 This is a supplementary section. I've posted a .pdf on our homework page.

Often the same DE can arise in completely different-looking situations. For example, first order linear DE's also arise (as special cases of second order linear DE's) in simple *RLC* circuit modeling.



circuit element	voltage drop	units
inductor	$L I'(t)$	L Henries (H)
resistor	$R I(t)$	R Ohms (Ω)
capacitor	$\frac{1}{C} Q(t)$	C Farads (F)

<http://cnx.org/content/m21475/latest/pic012.png>

Charge $Q(t)$ coulombs accumulates on the capacitor, at a rate $I(t)$ ($i(t)$ in the diagram above) amperes (coulombs/sec), i.e. $Q'(t) = I(t)$. $Q'' = I'$

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts). The units of voltage are energy units - Kirchoff's Law says that a test particle traversing any closed loop returns with the same potential energy level it started with:

For $Q(t)$: $L Q''(t) + R Q'(t) + \frac{1}{C} Q(t) = V(t)$

$\downarrow \frac{d}{dt}$

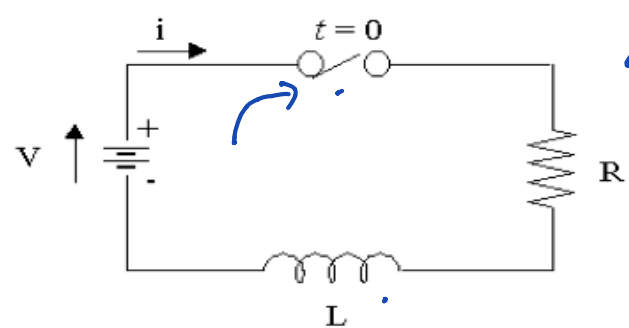
For $I(t)$: $L I''(t) + R I'(t) + \frac{1}{C} I(t) = V'(t)$

& Kirchoff

if no inductor, or if no capacitor, then Kirchoff's Law yields 1st order linear DE's, as below:

Exercise 3: Consider the $R - L$ circuit below, in which a switch is thrown at time $t = 0$. Assume the voltage V is constant, and $I(0) = 0$. Find $I(t)$. Interpret your results.

$L I'(t) + R I(t) = V$
 $\begin{cases} I'(t) + \frac{R}{L} I(t) = \frac{V}{L} \\ I(0) = 0 \end{cases}$



<http://www.intmath.com/differential-equations/5-rl-circuits.php>

$\begin{cases} x'(t) + a x(t) = b & a, b \text{ const.} \\ x(0) = x_0 \end{cases}$

$x(t) = \frac{b}{a} + (x_0 - \frac{b}{a}) e^{-at}$
 \uparrow
 const soln

transpose: $\frac{b}{a} = \frac{\frac{V}{L}}{\frac{R}{L}} = \frac{V}{R}$

$\Rightarrow I(t) = \frac{V}{R} + (0 - \frac{V}{R}) e^{-\frac{R}{L}t}$

$I(t) = \frac{V}{R} (1 - e^{-\frac{R}{L}t})$

learn in physics decays quickly

• finish 9.1.5, RLC circuits (Monday)
EP 3.7

• begin 2.1-2.2 (today's notes)
• Quiz: 1.4-1.5

Tues Jan 24

2.1 Improved population models

Let $P(t)$ be a population at time t . Let's call them "people", although they could be other biological organisms, decaying radioactive elements, accumulating dollars, or even molecules of solute dissolved in a liquid at time t (2.1.23). Consider:

$B(t)$, birth rate (e.g. $\frac{\text{people}}{\text{year}}$);

$\beta(t) := \frac{B(t)}{P(t)}$, fertility rate ($\frac{\text{people}}{\text{year}}$ per person)

$$B(t) = \beta(t)P(t)$$

$D(t)$, death rate (e.g. $\frac{\text{people}}{\text{year}}$);

$\delta(t) := \frac{D(t)}{P(t)}$, mortality rate ($\frac{\text{people}}{\text{year}}$ per person)

$$D(t) = \delta(t)P(t)$$

Then in a closed system (i.e. no migration in or out) we can write the governing DE two equivalent ways:

$$P'(t) = B(t) - D(t)$$

$$P'(t) = (\beta(t) - \delta(t))P(t)$$

Model 1: constant fertility and mortality rates, $\beta(t) \equiv \beta_0 \geq 0$, $\delta(t) \equiv \delta_0 \geq 0$, constants.

$$\Rightarrow P' = (\beta_0 - \delta_0)P = kP. \Rightarrow P(t) = P_0 e^{kt}$$

This is our familiar exponential growth/decay model, depending on whether $k > 0$ or $k < 0$.

Model 2: population fertility and mortality rates only depend on population P , but they are not constant:

$$\beta = \beta_0 + \beta_1 P$$

$$\delta = \delta_0 + \delta_1 P$$

with $\beta_0, \beta_1, \delta_0, \delta_1$ constants. This implies

$$P' = (\beta - \delta)P = ((\beta_0 + \beta_1 P) - (\delta_0 + \delta_1 P))P$$

$$P' = ((\beta_0 - \delta_0) + (\beta_1 - \delta_1)P)P$$

For viable populations, $\beta_0 > \delta_0$. For a sophisticated (e.g. human) population we might also expect

$\beta_1 < 0$, and resource limitations might imply $\delta_1 > 0$. With these assumptions, and writing $\beta_1 - \delta_1 = -a$, < 0 , $\beta_0 - \delta_0 = b > 0$ one obtains the logistic differential equation:

$$P' = (b - aP)P$$

$$P' = bP - aP^2, \text{ or equivalently}$$

$$P' = aP \left(\frac{b}{a} - P \right) = kP(M - P).$$

$k = a > 0, M = \frac{b}{a} > 0$. (One can consider other cases as well.)

$$\underline{P'(t) = kP(M - P)}$$

Exercise 1: Discuss qualitative features of the slope field for the logistic differential equation for $P = P(t)$:

$$P' = kP(M - P)$$

$$P > M \Rightarrow P'(t) = (+)(-) = (-)$$

$$P < M \Rightarrow P'(t) = (+)(+) = (+)$$

$\Rightarrow P(t)$ is decreasing

$$0 < P < M \Rightarrow P'(t) = (+)(+) = (+)$$

$\Rightarrow P(t)$ is increasing

$$P < 0 \Rightarrow P'(t) = (-)(+) = (-)$$

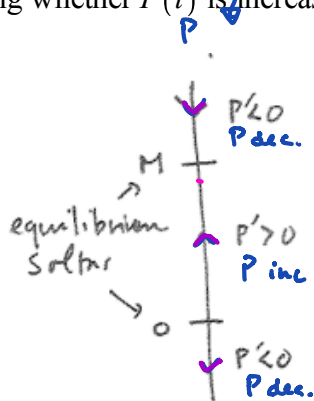
$\Rightarrow P(t)$ is decreasing

a) There are two constant ("equilibrium") solutions. What are they?

$$P = 0 \text{ const. soln.}$$

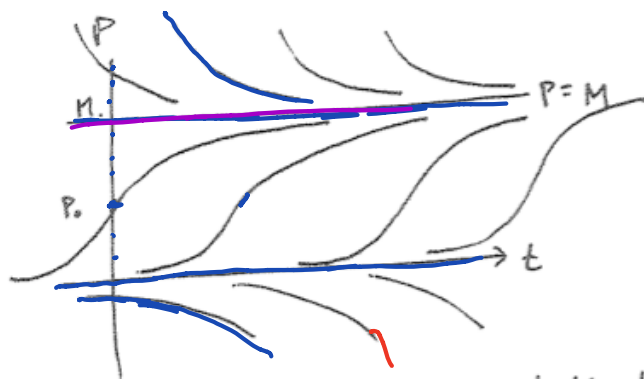
$$P = M \text{ const. soln.}$$

b) Evaluate the sign and magnitude of the slope function $f(P, t) = kP(M - P)$, in order to understand and be able to recreate the two diagrams below. One is a qualitative picture of the slope field, in the $t - P$ plane. The diagram to the left of it, called the phase diagram, is just a P number line with arrows indicating whether $P(t)$ is increasing or decreasing on the intervals between the constant solutions.

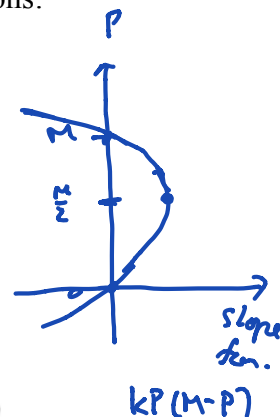


phase portrait

wed start here. diagram



slope field, as represented by typical solution graphs



if any of solution graphs touched at common point (t_1, P_1) then IVP there would not have a unique solution. $kMP - kP^2$ $f(t, P) = kP(M - P)$ $\frac{\partial f}{\partial P} = kM - 2kP$ $\text{cont in } (t, P)$ $\text{cont in } (t, P)$

c) When discussing the logistic equation, the value M is called the "carrying capacity" of the (ecological or other) system. Discuss why this is a good way to describe M . Hint: if $P(0) = P_0 > 0$, and $P(t)$ solves the logistic equation, what is the apparent value of $\lim_{t \rightarrow \infty} P(t)$?

M carrying capacity

solution unique

example

L >

$$\begin{cases} x'(t) = 3x(x-5) \\ x(0) = 2 \end{cases}$$

separate:

$$\int \frac{dx}{x(x-5)} = \int 3 dt \quad (x \neq 0, 5)$$

$$5 \int \frac{1}{5} \left(\frac{1}{x-5} - \frac{1}{x} \right) dx = 5 \int 3 dt$$

$$\int \frac{1}{x-5} - \frac{1}{x} dx = \int 15 dt$$

$$\ln|x-5| - \ln|x| = 15t + C_1$$

$$e^{\ln \left| \frac{x-5}{x} \right|} = e^{15t + C_1}$$

$$\left| \frac{x-5}{x} \right| = e^{15t} e^{C_1}$$

$$\frac{x-5}{x} = C e^{15t} \quad C = \pm e^{C_1}$$

$$\text{@ } t=0: -\frac{3}{2} = C e^0 = C.$$

$$x(0) = 2$$

$$x \cdot \frac{x-5}{x} = -\frac{3}{2} e^{15t} \cdot x$$

$$x-5 = x \left(-\frac{3}{2} e^{15t} \right)$$

$$x + \frac{3}{2} e^{15t} x = 5$$

$$x \left(1 + \frac{3}{2} e^{15t} \right) = 5$$

$$x = \frac{5}{1 + \frac{3}{2} e^{15t}} \quad \frac{2}{2} = \frac{10}{2 + 3e^{15t}}$$

partial fractions

$$\frac{1}{x(x-5)} = \frac{A}{x} + \frac{B}{x-5}$$

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

shortcut:

$$\cdot \frac{1}{5} \left(\frac{1}{x-5} - \frac{1}{x} \right)$$

$$\frac{1}{5} \frac{x - (x-5)}{x(x-5)}$$

$$\frac{1}{5} \frac{5}{x(x-5)}$$

$$= \frac{1}{5} \left(\frac{1}{x-5} - \frac{1}{x} \right)$$

long way:

mult both sides by $x(x-5)$

$$1 = A(x-5) + Bx$$

$$\text{@ } x=0: 1 = -5A \Rightarrow A = -\frac{1}{5}$$

$$\text{@ } x=5: 1 = 5B \Rightarrow B = \frac{1}{5}$$

$$\frac{1}{x(x-5)} = -\frac{1}{5} \frac{1}{x} + \frac{1}{5} \frac{1}{x-5}$$

shortcut

Exercise 2: Solve the logistic DE IVP

$$\begin{aligned} P' &= k P (M - P) \\ P(0) &= P_0 \end{aligned}$$

via separation of variables. Verify that the solution formula is consistent with the slope field and phase diagram discussion from exercise 1. Hint: You should find that

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}.$$

Solution (we will work this out step by step in class):

$$\frac{dP}{P(P - M)} = -k dt$$

By partial fractions,

$$\frac{1}{P(P - M)} = \frac{1}{M} \left(\frac{1}{P - M} - \frac{1}{P} \right).$$

Use this expansion and multiply both sides of the separated DE by M to obtain

$$\left(\frac{1}{P - M} - \frac{1}{P} \right) dP = -kM dt.$$

Integrate:

$$\begin{aligned} \ln|P - M| - \ln|P| &= -Mkt + C_1 \\ \ln \left| \frac{P - M}{P} \right| &= -Mkt + C_1 \end{aligned}$$

exponentiate:

$$\left| \frac{P - M}{P} \right| = C_2 e^{-Mkt}$$

Since the left-side is continuous

$$\frac{P - M}{P} = C e^{-Mkt} \quad (C = C_2 \text{ or } C = -C_2)$$

(At $t = 0$ we see that

$$\frac{P_0 - M}{P_0} = C.)$$

Now, solve for $P(t)$ by multiplying both sides of the second to last equation by $P(t)$:

$$P - M = C e^{-Mkt} P$$

Collect $P(t)$ terms on left, and add M to both sides:

$$\begin{aligned} P - C e^{-Mkt} P &= M \\ P(1 - C e^{-Mkt}) &= M \\ P &= \frac{M}{1 - C e^{-Mkt}}. \end{aligned}$$

Plug in C and simplify:

$$P = \frac{M}{1 - \left(\frac{P_0 - M}{P_0} \right) e^{-Mkt}} = \frac{MP_0}{P_0 - (P_0 - M)e^{-Mkt}}$$

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-Mkt}}$$

Finally, because $\lim_{t \rightarrow \infty} e^{-Mkt} = 0$, we see that

$$\lim_{t \rightarrow \infty} P(t) = \frac{MP_0}{P_0} = M \text{ as expected.}$$

$$\lim_{t \rightarrow \infty} e^{-Mkt} = 0$$

solution
formula verifies
the phase diagram
& slope field pictures

Note: If $P_0 > 0$ the denominator stays positive for $t \geq 0$, so we know that the formula for $P(t)$ is a differentiable function for all $t > 0$. (If the denominator became zero, the function would blow up at the corresponding vertical asymptote.) To check that the denominator stays positive check that (i) if $P_0 < M$ then the denominator is a sum of two positive terms; if $P_0 = M$ the separation algorithm actually fails because you divided by 0 to get started but the formula actually recovers the constant equilibrium solution $P(t) \equiv M$; and if $P_0 > M$ then $|M - P_0| < P_0$ so the second term in the denominator can never be negative enough to cancel out the positive P_0 , for $t > 0$.)

Question: You have a couple of homework problems where you are asked to find solutions $x(t)$ to differential equations of the form

$$x'(t) = a(x - b) \cdot (x - c).$$

How would you proceed?

$$\frac{dx}{dt} = a(x - b)(x - c)$$

$$\int \frac{dx}{(x - b)(x - c)} = \int a dt$$

$$\int \frac{A}{x - b} + \frac{B}{x - c} dx = \int a dt$$

$$\frac{1}{(x - b)(x - c)} = \frac{A}{x - b} + \frac{B}{x - c}$$

$$A \ln|x - b| + B \ln|x - c| = at + C$$

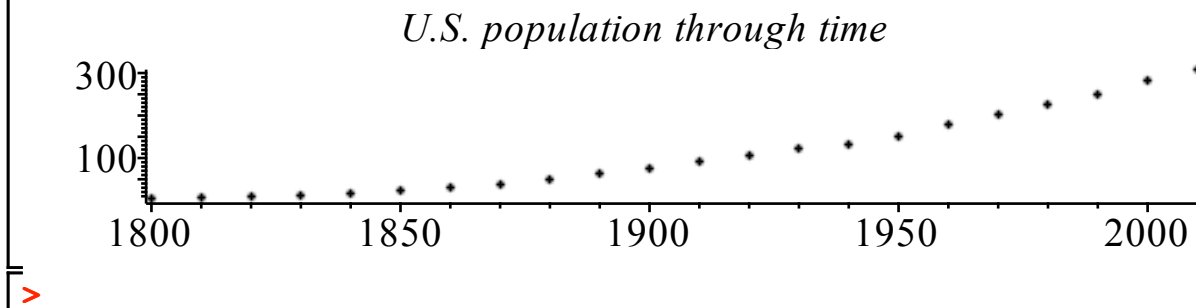
$$e^{A \ln|x - b| + B \ln|x - c|} = e^{at} e^C$$

...

Application!

The Belgian demographer P.F. Verhulst introduced the logistic model around 1840, as a tool for studying human population growth. Our text demonstrates its superiority to the simple exponential growth model, and also illustrates why mathematical modelers must always exercise care, by comparing the two models to actual U.S. population data.

```
> restart : # clear memory
  Digits := 5 : #work with 5 significant digits
> pops := [[1800, 5.3], [1810, 7.2], [1820, 9.6], [1830, 12.9],
  [1840, 17.1], [1850, 23.2], [1860, 31.4], [1870, 38.6],
  [1880, 50.2], [1890, 63.0], [1900, 76.2], [1910, 92.2],
  [1920, 106.0], [1930, 123.2], [1940, 132.2], [1950, 151.3],
  [1960, 179.3], [1970, 203.3], [1980, 225.6], [1990, 248.7],
  [2000, 281.4], [2010, 308.]] : #I added 2010 - between 306-313
  # I used shift-enter to enter more than one line of information
  # before executing the command.
> with(plots) : # plotting library of commands
  pointplot(pops, title = 'U.S. population through time');
```



Unlike Verhulst, the book uses data from 1800, 1850 and 1900 to get constants in our two models. We let $t=0$ correspond to 1800.

$$t=0$$
$$P_0 = 5.3$$

Exponential Model: For the exponential growth model $P(t) = P_0 e^{rt}$ we use the 1800 and 1900 data to get values for P_0 and r :

$$P(100) = 76.2 \text{ million.}$$

```
> P0 := 5.308;
  solve(P0·exp(r·100) = 76.212, r);
  P0 := 5.308
  0.026643 (2)
> P1 := t→5.308·exp(.02664·t); #exponential model -eqtn (9) page 83
  P1 := t→5.308 e0.02664 t (3)
```

Logistic Model: We get P_0 from 1800, and use the 1850 and 1900 data to find k and M :

> $P2 := t \rightarrow M \cdot P0 / (P0 + (M - P0) \cdot \exp(-M \cdot k \cdot t));$ # logistic solution we worked out

$$P2 := t \rightarrow \frac{M P0}{P0 + (M - P0) e^{-M k t}} \quad (4)$$

> $\text{solve}(\{P2(50) = 23.192, P2(100) = 76.212\}, \{M, k\});$
 $\{M = 188.12, k = 0.00016772\}$ (5)

> $M := 188.12;$
 $k := .16772e-3;$
 $P2(t);$ #should be our logistic model function,
 #equation (11) page 84.

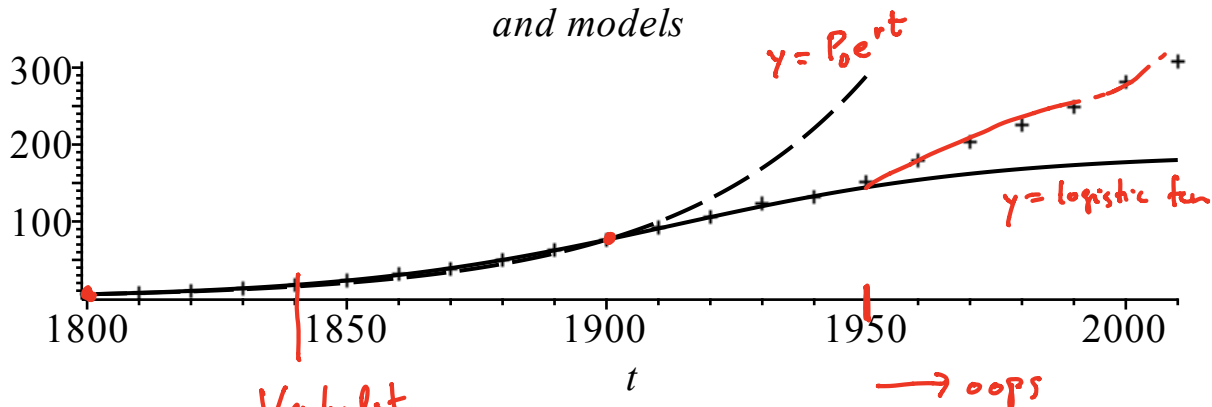
$$\frac{M := 188.12}{k := 0.00016772} \quad (6)$$

$$\frac{998.54}{5.308 + 182.81 e^{-0.031551 t}}$$

> Now compare the two models with the real data, and discuss. The exponential model takes no account of the fact that the U.S. has only finite resources. Any ideas on why the logistic model begins to fail (with our parameters) around 1950?

> $\text{plot1} := \text{plot}(P1(t-1800), t = 1800..1950, \text{color} = \text{black}, \text{linestyle} = 3);$
 #this linestyle gives dashes for the exponential curve
 $\text{plot2} := \text{plot}(P2(t-1800), t = 1800..2010, \text{color} = \text{black});$
 $\text{plot3} := \text{pointplot}(\text{pops}, \text{symbol} = \text{cross});$
 $\text{display}(\{\text{plot1}, \text{plot2}, \text{plot3}\}, \text{title} = \text{'U.S. population data and models'});$

U.S. population data
and models



baby boom.

better resources — fertility rates went up.
 longer lives — mortality rates went down in mid 1900's.

- HW hand-in & pick-up up front
- quit at end of class
- finish §2.1
- start §2.2 pages 1-3 today's notes

Math 2250-004
Wednesday Jan 25

2.2: Autonomous Differential Equations.

Recall, that if we solve for the derivative, a general first order DE for $x = x(t)$ is written as

$$x' = f(t, x),$$

slope fun depends on x & t

which is shorthand for $x'(t) = f(t, x(t))$.

Definition: If the slope function f only depends on the value of $x(t)$, and not on t itself, then we call the first order differential equation autonomous:

$$x' = f(x).$$

(rate of change) $P'(t) = kP(M-P)$
slope only depends on x

Example: The logistic DE, $P' = kP(M - P)$ is an autonomous differential equation for $P(t)$, for example.

Definition: Constant solutions $x(t) \equiv c$ to autonomous differential equations $x' = f(x)$ are called equilibrium solutions. Since the derivative of a constant function $x(t) \equiv c$ is zero, the values c of equilibrium solutions are exactly the roots c to $f(c) = 0$.

$x(t) \equiv c$ solution

$$x'(t) = 0 = f(x) = f(c)$$

Example: The functions $P(t) \equiv 0$ and $P(t) \equiv M$ are the equilibrium solutions for the logistic DE.

Exercise 1: Find the equilibrium solutions of

1a) $x'(t) = 3x - x^2 = x(3 - x)$

equil solns are $x(t) \equiv 0$
 $x(t) \equiv 3$

1b) $x'(t) = x^3 + 2x^2 + x = x(x^2 + 2x + 1)$
 $= x(x+1)^2$

equil solns $x \equiv 0$
 $x \equiv -1$

1c) $x'(t) = \sin(x)$.

equilibrium solns

$$x \equiv 0$$

$$x = \pi n, n \in \mathbb{Z}$$

↑
integers.

$$x = 0, \pm\pi, \pm2\pi, \dots$$

"start close, stay close"

Def: Let $x(t) \equiv c$ be an equilibrium solution for an autonomous DE. Then

c is a stable equilibrium solution if solutions with initial values close enough to c stay close to c .

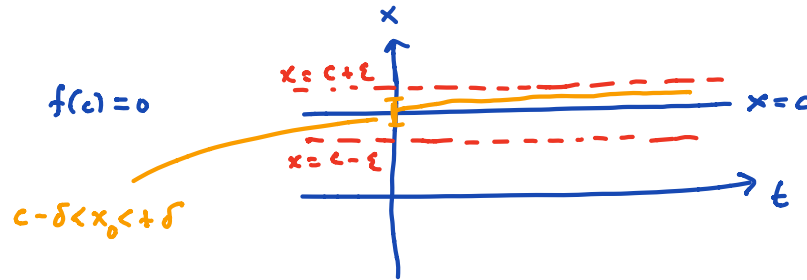
for later time

There is a precise way to say this, but it requires quantifiers: For every $\epsilon > 0$ there exists a $\delta > 0$ so that for solutions with $|x(0) - c| < \delta$, we have $|x(t) - c| < \epsilon$ for all $t > 0$.

c is an unstable equilibrium if it is not stable.

c is an asymptotically stable equilibrium solution if it's stable and in addition, if $x(0)$ is close enough to c , then $\lim_{t \rightarrow \infty} x(t) = c$, i.e. there exists a $\delta > 0$ so that if $|x(0) - c| < \delta$ then

$\lim_{t \rightarrow \infty} x(t) = c$. (Notice that this means the horizontal line $x = c$ will be an asymptote to the solution graphs $x = x(t)$ in these cases.)

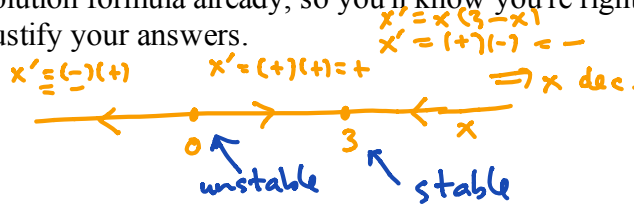


Exercise 2: Use phase diagram analysis to guess the stability of the equilibrium solutions in Exercise 1.

For (a) you've worked out a solution formula already, so you'll know you're right. For (b), (c), use the Theorem on the next page to justify your answers.

2a) $x'(t) = 3x - x^2$

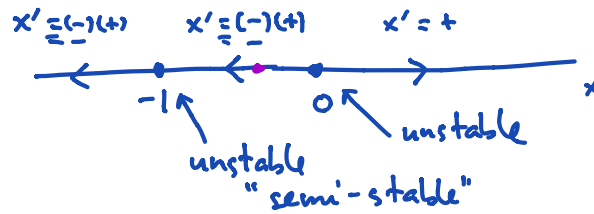
$x(3-x)$



($x=3, 0$ equil solns)
($x=3, 0$ equil. pts.)

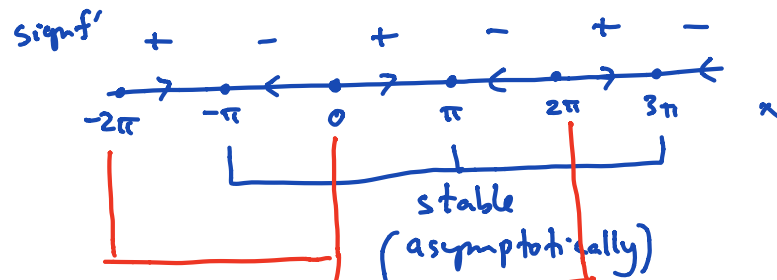
2b) $x'(t) = x^3 + 2x^2 + x$

$= x(x+1)^2$



2c) $x'(t) = \sin(x)$

$x = n\pi$



even multiples of π are unstable

fact $-1 < x_0 < 0$
then $\lim_{t \rightarrow \infty} x(t) = -1$
 $x_0 < -1$, then $x(t) \rightarrow -\infty$
either in infinite or finite time.

Theorem: Consider the autonomous differential equation

$$x'(t) = f(x)$$

with $f(x)$ and $\frac{\partial}{\partial x} f(x)$ continuous (so local existence and uniqueness theorems hold). Let $f(c) = 0$, i.e.

$x(t) \equiv c$ is an equilibrium solution. Suppose c is an *isolated zero* of f , i.e. there is an open interval containing c so that c is the only zero of f in that interval. The the stability of the equilibrium solution c can be completely determined by the local phase diagrams:

$$\begin{array}{llll} \text{sign}(f) : - - - 0 + + + & \Rightarrow & \begin{array}{c} \leftarrow \leftarrow \leftarrow c \rightarrow \rightarrow \rightarrow \\ \leftarrow \leftarrow \leftarrow c \rightarrow \rightarrow \rightarrow \end{array} & \Rightarrow c \text{ is unstable} \\ \text{sign}(f) : + + + 0 - - - & \Rightarrow & \begin{array}{c} \rightarrow \rightarrow \rightarrow c \leftarrow \leftarrow \leftarrow \\ \rightarrow \rightarrow \rightarrow c \leftarrow \leftarrow \leftarrow \end{array} & \Rightarrow c \text{ is asymptotically stable} \\ \text{sign}(f) : + + + 0 + + + & \Rightarrow & \begin{array}{c} \rightarrow \rightarrow \rightarrow c \rightarrow \rightarrow \rightarrow \\ \rightarrow \rightarrow \rightarrow c \rightarrow \rightarrow \rightarrow \end{array} & \Rightarrow c \text{ is unstable (half stable)} \\ \text{sign}(f) : - - - 0 - - - & \Rightarrow & \begin{array}{c} \leftarrow \leftarrow \leftarrow c \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow c \leftarrow \leftarrow \leftarrow \end{array} & \Rightarrow c \text{ is unstable (half stable)} \end{array}$$

You can actually prove this Theorem with calculus!! (want to try?)

$$\begin{array}{c} f' > 0 \quad f' < 0 \\ \sim f' < 0 \quad \text{or } f' < 0 \\ \hline c \\ f(c) = 0 \end{array}$$

Here's why!

e.g. consider the second case

$$\begin{array}{c} f > 0 \quad f < 0 \\ \hline \rightarrow \quad \leftarrow \\ \hline f > 0 \quad f < 0 \\ \hline [a, b] \quad c \\ a \quad x_0 \quad b \quad c \end{array}$$

f cont; $f > 0$ on subinterval $[a, b]$

$$\Rightarrow f \geq \delta > 0 \text{ on } [a, b]$$

(extreme value thm from calculus, f attains its minimum).

$$\Rightarrow x'(t) \geq \delta \text{ as long as } x(t) \in [a, b]$$

$\Rightarrow x(t)$ stays in this interval for time interval at most $\frac{b-a}{\delta}$

$$\begin{array}{l} x'(t) = f(x(t)) \\ \geq \delta \end{array}$$

Exercise 3) Use the chain rule to check that if $x(t)$ solves the autonomous DE

$$x'(t) = f(x)$$

Then $X(t) := x(t - c)$ solves the same DE. What does this say about the geometry of representative solution graphs to autonomous DEs? Have we already noticed this?

horizontal translations of solution graphs are solution graphs

$$x'(r) = f(x(r))$$

filled in after class:

$$\frac{d}{dt} X(t) = x'(t-c) \cdot 1 \quad (\text{chain rule}); \text{ but } x'(t-c) = f(x(t-c)) \text{ DE} \\ = f(X(t)) \quad \checkmark$$

Further application: Doomsday-extinction. With different hypotheses about fertility and mortality rates, one can arrive at a population model which looks like logistic, except the right hand side is the opposite of what it was in that case:

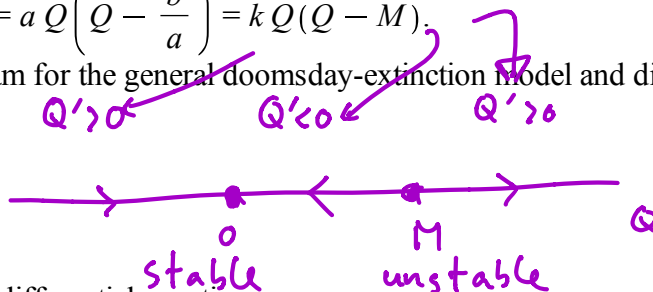
$$\text{Logistic:} \quad P'(t) = -aP^2 + bP$$

$$\text{Doomsday-extinction:} \quad Q'(t) = aQ^2 - bQ$$

For example, suppose that the chances of procreation are proportional to population density (think alligators), i.e. the fertility rate $\beta = aQ(t)$, where $Q(t)$ is the population at time t . Suppose the morbidity rate is constant, $\delta = b$. With these assumptions the birth and death rates are aQ^2 and $-bQ$ which yields the DE above. In this case factor the right side:

$$Q'(t) = aQ \left(Q - \frac{b}{a} \right) = kQ(Q - M).$$

Exercise 4a) Construct the phase diagram for the general doomsday-extinction model and discuss the stability of the equilibrium solutions.



Exercise 4b) If $P(t)$ solves the logistic differential equation

$$P'(t) = kP(M - P)$$

show that $Q(t) := P(-t)$ solves the doomsday-extinction differential equation

$$Q'(t) = kQ(Q - M).$$

Use this to recover a formula for solutions to doomsday-extinction IVPs. What does this say about how representative solution graphs are related, for the logistic and the doomsday-extinction models? Recall, the solution to the logistic IVP is

$$P(t) = \frac{MP_0}{(M - P_0)e^{-Mkt} + P_0}.$$

using the general fact on the right,

$$Q(t) = P(-t) = \frac{MP_0}{(M - P_0)e^{Mkt} + P_0}$$

Satisfies

$$\begin{cases} Q'(t) = kQ(Q - M) \\ Q(0) = P_0 \end{cases}$$

If $x(t)$ solves

$$x'(t) = f(x(t))$$

then $z(t) = x(-t)$

$$\text{solves } z'(t) = -f(z(t))$$

check:

$$z'(t) = x'(-t) \cdot (-1)$$

$$= f(x(-t))(-1)$$

$$= -f(z(t))$$

Exercise 5) Use your formula from the previous exercise to transcribe the solution to the doomsday-extinction IVP

$$\begin{aligned}x'(t) &= x(x-1) \\ x(0) &= 2.\end{aligned}$$

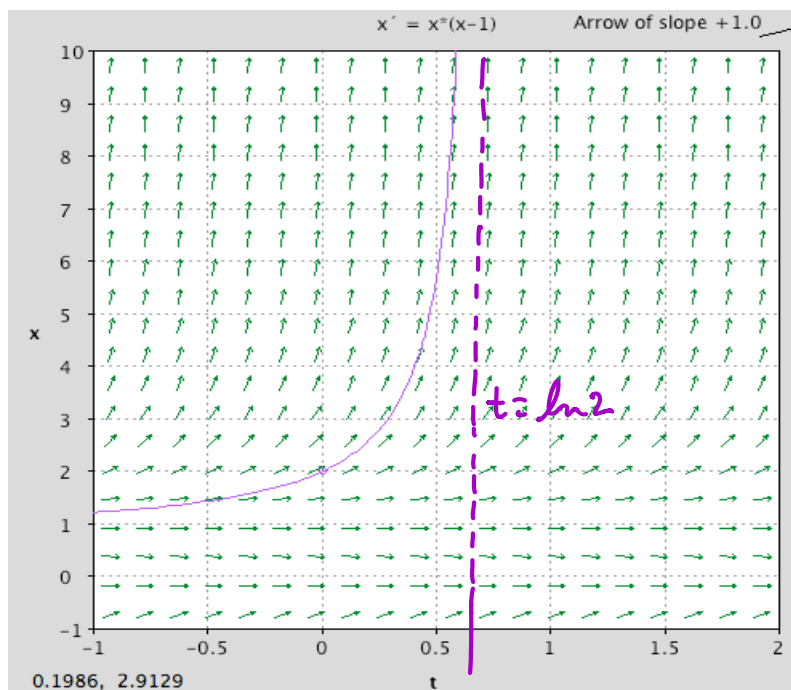
Does the solution exist for all $t > 0$? (Hint: no, there is a very bad doomsday at $t = \ln 2$.)

use formula on previous page. $M=1, k=1, P_0=2$

$$x(t) = \frac{2}{(1-2)e^t + 2} = \frac{2}{-e^t + 2}$$

$$-e^t + 2 = 0 \Leftrightarrow t = \ln 2. \quad \lim_{t \rightarrow \ln 2^-} x(t) = \frac{2}{+0} = +\infty$$

"doomsday"
@ $t = \ln 2$



$$kP(M-P)$$

Further application: harvesting a logistic population... text p.97 (or, why do fisheries sometimes seem to die out "suddenly"?). Consider the DE

$$P'(t) = aP - bP^2 - h$$

Notice that the first two terms represent a logistic rate of change, but we are now harvesting the population at a rate of h units per time. For simplicity we'll assume we're harvesting fish per year (or thousands of fish per year etc.) One could model different situations, e.g. constant "effort" harvesting, in which the effect on how fast the population was changing could be hP instead of P .

For computational ease we will assume $a = 2, b = 1$. (One could actually change units of population and time to reduce to this case.)

$$P'(t) = 2P - P^2 - h$$

$$P'(t) = -P^2 + 2P - h$$

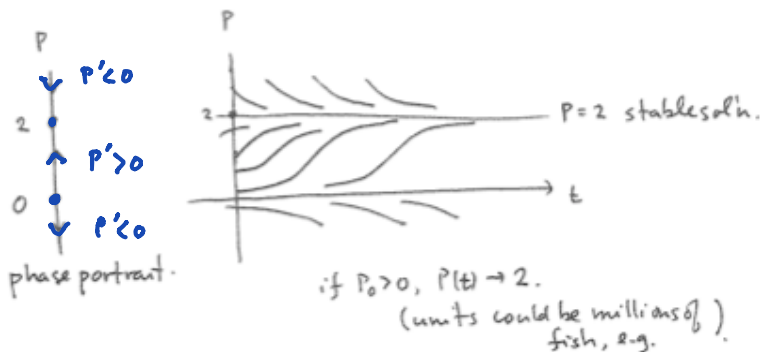
for computational simplicity
take $a = 2, b = 1$

Case 0 no harvesting

$$P'(t) = 2P - P^2$$

$$= P(2 - P)$$

equilibrium
 $P = 2, 0$



with harvesting:

$$P'(t) = 2P - P^2 - h$$

$$= - (P^2 - 2P + h)$$

$$= - (P - P_1)(P - P_2)$$

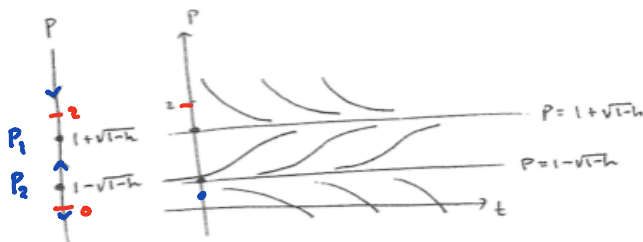
$$P_1, P_2 = \frac{2 \pm \sqrt{4 - 4h}}{2}$$

$$P_1, P_2 = 1 \pm \sqrt{1 - h}$$

Case 1: subcritical harvesting

$$0 < h < 1$$

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Case 2: Critical harvesting

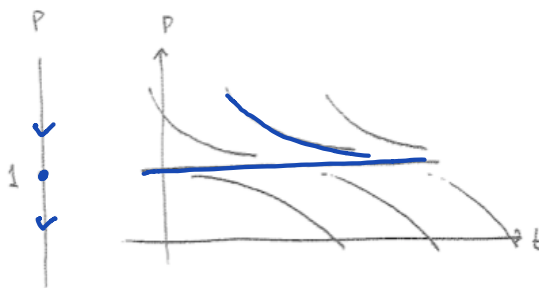
$$h = 1$$

$$P'(t) = - (P - 1)^2$$

$$= 2P - P^2 - 1$$

$$= - (P^2 - 2P + 1)$$

$$= - (P - 1)^2$$

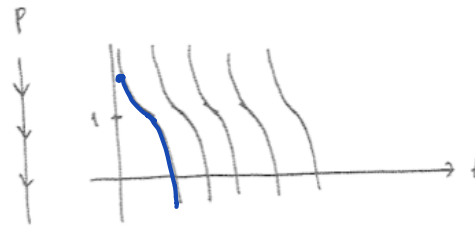


Case 3 Overharvesting

$$h > 1$$

complex roots.

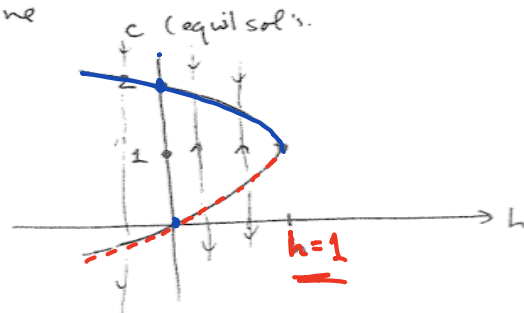
$$\begin{aligned} P'(t) &= -(P^2 - 2P + h) \\ &= -[(P-1)^2 + (h-1)] \\ &< 0. \end{aligned}$$



This model gives a plausible explanation for why many fisheries have "unexpectedly" collapsed in modern history. If $h < 1$ but near 1 and something perturbs the system a little bit (a bad winter, or a slight increase in fishing pressure), then the population and/or model could suddenly shift so that $P(t) \rightarrow 0$ very quickly.

Here's one picture that summarizes all the cases - you can think of it as collection of the phase diagrams for different fishing pressures h . The upper half of the parabola represents the stable equilibria, and the lower half represents the unstable equilibria. Diagrams like this are called "bifurcation diagrams". In the sketch below, the point on the h -axis should be labeled $h = 1$, not h . What's shown is the parabola of equilibrium solutions, $c = 1 \pm \sqrt{1-h}$, i.e. $2c - c^2 - h = 0$, i.e. $h = c(2-c)$.

"bifurcation diagram" of equilibrium solutions in the h - c plane



• We'll finish "Wed." notes today and do "Fri." notes on Monday §2.1-2.2

Math 2250-004

Fri Jan 27 30
Ma

2.3 Improved velocity models: velocity-dependent drag forces

For particle motion along a line, with

$$\begin{aligned} &\text{position } x(t) \text{ (or } y(t) \text{) ,} \\ &\text{velocity } x'(t) = v(t) \text{ , and} \\ &\text{acceleration } x''(t) = v'(t) = a(t) \end{aligned}$$

We have Newton's 2nd law

$$m v'(t) = F$$

where F is the net force.

- We're very familiar with constant force $F = m \alpha$, where α is a constant:

$$\begin{aligned} v'(t) &= \alpha \\ v(t) &= \alpha t + v_0 \\ x(t) &= \frac{1}{2} \alpha t^2 + v_0 t + x_0 . \end{aligned}$$

Examples we've seen a lot of:

- $\alpha = -g$ near the surface of the earth, if up is the positive direction, or $\alpha = g$ if down is the positive direction.
- boats or cars or "particles" subject to constant acceleration or deceleration.

New today !!! Combine a constant force with a velocity-dependent drag force, at the same time. The text calls this a "resistance" force:

$$m v'(t) = m \alpha + F_R$$

Empirically/mathematically the resistance forces F_R depend on velocity, in such a way that their magnitude is

$$|F_R| \approx k |v|^p , 1 \leq p \leq 2 .$$

- $p = 1$ (linear model, drag proportional to velocity):

$$m v'(t) = m \alpha - k v$$

This linear model makes sense for "slow" velocities, as a linearization of the frictional force function, assuming that the force function is differentiable with respect to velocity...recall Taylor series for how the velocity resistance force might depend on velocity:

$$F_R(v) = F_R(0) + F_R'(0) v + \frac{1}{2!} F_R''(0) v^2 + \dots$$

$F_R(0) = 0$ and for small enough v the higher order terms might be negligible compared to the linear term, so

$$F_R(v) \approx F_R'(0) v \approx -k v .$$

We write $-k v$ with $k > 0$, since the frictional force opposes the direction of motion, so sign opposite of the velocity's.

[http://en.wikipedia.org/wiki/Drag_\(physics\)#Very_low_Reynolds_numbers:_Stokes.27_drag](http://en.wikipedia.org/wiki/Drag_(physics)#Very_low_Reynolds_numbers:_Stokes.27_drag)

Exercise 1a: Rewrite the linear drag model as

$$v'(t) = \alpha - \rho v$$

where the $\rho = \frac{k}{m}$. Construct the phase diagram for v . Notice that $v(t)$ has exactly one constant (equilibrium) solution, and find it. Its value is called the *terminal velocity*. Explain why *terminal velocity* is an appropriate term of art, based on your phase diagram.

1b) Solve the IVP

$$\begin{aligned} v'(t) &= \alpha - \rho v \\ v(0) &= v_0 \end{aligned}$$

and verify your phase diagram analysis. (This is, once again, our friend the first order constant coefficient linear differential equation.)

1c) integrate the velocity function above to find a formula for the position function $y(t)$.

- $p = 2$, for the power in the resistance force. This can be an appropriate model for velocities which are not "near" zero....described in terms of "Reynolds number". Accounting for the fact that the resistance opposes direction of motion we get

$$\begin{aligned} m v'(t) &= m \alpha - k v^2 & \text{if } v > 0 \\ m v'(t) &= m \alpha + k v^2 & \text{if } v < 0. \end{aligned}$$

[http://en.wikipedia.org/wiki/Drag_\(physics\)#Drag_at_high_velocity](http://en.wikipedia.org/wiki/Drag_(physics)#Drag_at_high_velocity)

Exercise 2) Once again letting $\rho = \frac{k}{m}$ we can rewrite the DE's as

$$\begin{aligned} v'(t) &= \alpha - \rho v^2 & \text{if } v > 0 \\ v'(t) &= \alpha + \rho v^2 & \text{if } v < 0. \end{aligned}$$

2a) Consider the case in which $\alpha = -g$, so we are considering vertical motion, with up being the positive direction. Draw the phase diagrams. Note that each diagram contains a half line of v -values. Make conclusions about velocity behavior in case $v_0 > 0$ and $v_0 \leq 0$. Is there a terminal velocity?

2b) Set up the two separable differential equation IVPs for the cases above, so that you will be able to complete finding the solution formulas (in your homework)....Of course, once you find the velocity function you'll still need to integrate that, if you want to find the position function!

Application: We consider the bow and deadbolt example from the text, page 102-104. It's shot vertically into the air (watch out below!), with an initial velocity of $49 \frac{m}{s}$. In the no-drag case, this could just be the vertical component of a deadbolt shot at an angle. With drag, one would need to study a more complicated system of DE's for the horizontal and vertical motions, if you didn't shoot the bolt straight up.

Exercise 3: First consider the case of no drag, so the governing equations are

$$\begin{aligned}v'(t) &= -g \approx -9.8 \frac{m}{s^2} \\v(t) &= -g t + v_0 = -g t + 49 g \\x(t) &= -\frac{1}{2} g t^2 + v_0 t + x_0 = -\frac{1}{2} g t^2 + 49 g t.\end{aligned}$$

Find when $v = 0$ and deduce how long the object rises, how long it falls, and its maximum height.

Maple check:

```
[> restart :
  Digits := 5 :

> g := 9.8;
  v0 := 49.0;
  v1 := t -> -g*t + v0;
  y1 := t -> -1/2*g*t^2 + v0*t;

                                g := 9.8
                                v0 := 49.0
                                v1 := t -> -g*t + v0
                                y1 := t -> -1/2*g*t^2 + v0*t
```

(7)

Exercise 4: Now consider the linear drag model for the same deadbolt, with the same initial velocity of $5g = 49 \frac{m}{s}$. We'll assume that our deadbolt has a measured terminal velocity of $v_\tau = -245 \frac{m}{s} = -25g$,

so $|v_\tau| = 25g = \frac{g}{\rho} \Rightarrow \rho = .04$ (convenient). So, from our earlier work:

$$v = v_\tau + (v_0 - v_\tau) e^{-\rho t}$$

$$y = y_0 + t v_\tau + \frac{(v_0 - v_\tau)(1 - e^{-\rho t})}{\rho}$$

So,

$$v = -\frac{g}{\rho} + \left(v_0 + \frac{g}{\rho}\right) e^{-\rho t} = -245 + 294 e^{-.04 t}.$$

$$y = 0 - 245 t + \frac{294}{.04} (1 - e^{-.04 t}).$$

When does the object reach its maximum height, what is this height, and how long does the object fall? Compare to the no-drag case with the same initial velocity, in Exercise 3.

Maple check, and then work:

```
[>
with(DEtools):
> g := 9.8; rho := .04; v0 := 49;
                                g := 9.8
                                rho := 0.04
                                v0 := 49
(8)

> dsolve({v'(t) = -g - rho*v(t), v(0) = v0}, v(t));
                                v(t) = -245 + 294 e^(-1/25 t)
(9)

> v2 := t -> -245.0 + 294 e^(-1/25 t);
                                v2 := t -> -245.0 + 294 e^(-1/25 t)
(10)

> solve(v2(t) = 0, t);
                                4.5580
(11)

> y2 := t -> y0 + int(-g/rho + e^(-rho*s) (v0 + g/rho) ds,
                                y2 := t -> y0 + int((-g/rho + e^(-rho*s) (v0 + g/rho)) ds
(12)
```

```

> v2(t);
  294
  .04
;
y0 := 0;
y2(t);

```

$$-245.0 + 294 e^{-\frac{1}{25} t}$$

$$7350.0$$

$$y0 := 0$$

$$7350. - 245. t - 7350. e^{-0.040000 t}$$

(13)

```

> solve(v2(t) = 0, t);
solve(y2(t) = 0, t);
y2(4.558);

```

$$4.5580$$

$$9.4110, 0.$$

$$108.28$$

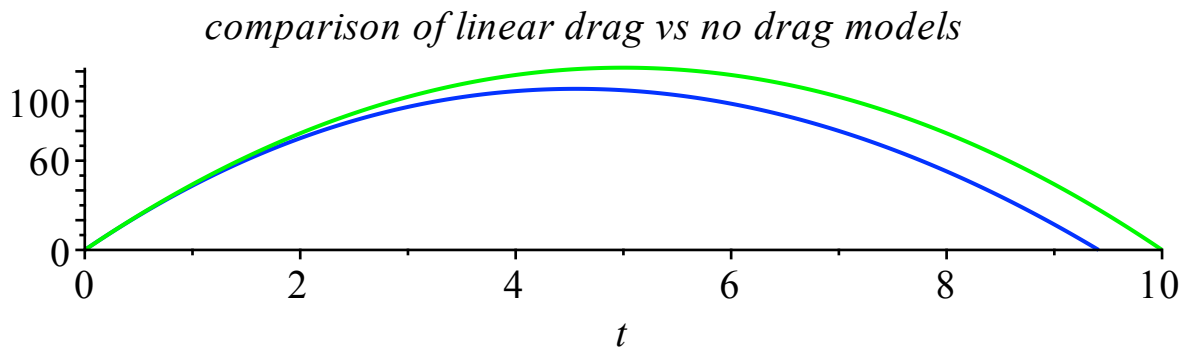
(14)

picture:

```

> with(plots) :
plot1 := plot(y1(t), t = 0..10, color = green) :
plot2 := plot(y2(t), t = 0..9.4110, color = blue) :
display( {plot1, plot2}, title = `comparison of linear drag vs no drag models`);

```



```

>

```