Math 2250-004
Fri Jan 20
1.5: linear DEs, and applications.

· postpone experiment.
– so we can talk about §1.5.
· next week's notes are posted
· printing is free in Math tutoring Ctr. basement JWB, LCB

Section 1.5, linear differential equations: → because y', y appear to the 1^{st} power, possibly multiplied by a fcn of x

A first order linear DE for $y(x)$ is one that can be written as

$$y' + P(x)y = Q(x)$$

or if $x = x(t)$ $\quad x'(t) + P(t)x(t) = Q(t)$

Exercise 1: Classify the differential equations below as linear, separable, both, or neither. Justify your answers.

$x' + P(t)x = Q(t)$

$\left(\frac{dy}{dx} = f(x)g(y)\right)$ sep

a) $y'(x) = -2y + 4x^2$
b) $y'(x) = x - y^2 + 1$ bad y^2
c) $y'(x) = x^2 - x^2 y + 1$
d) $y'(x) = \dfrac{6x - 3xy}{x^2+1}$
e) $y'(x) = x^2 + y^2$
f) $y'(x) = x^2 e^{x^3}$. §1.2

	linear	separable
a)	✓	no
b)	no	no
c)	✓	no
d)	✓	✓
e)	no	no
f)	✓	✓

a) $y' + \overbrace{2}^{P(x)}y = \overbrace{4x^2}^{Q(x)}$

b) can't have a y^2 term
must be y^1 for linear

c) $y' + x^2 y = x^2 + 1$

d) $y' + \overbrace{\dfrac{3x}{x^2+1}}^{P(x)}\, y = \dfrac{6x}{x^2+1}$ linear

$\dfrac{dy}{dx} = \dfrac{3x}{x^2+1}(2-y)$ sep.

or $\dfrac{x}{x^2+1}(6-3y)$

e)

f) $y' + 0y = x^2 e^{x^3}$ $(P(x) \equiv 0)$ $\quad \dfrac{dy}{dx} = x^2 e^{x^3} \cdot 1$

Algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)dx$ be any antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE: integrating factor I.F.

$$\overbrace{e^{\int P(x)dx}}^{f'} \left(y' + P(x)y\right) = e^{\int P(x)dx} Q(x)$$

. equivalent to

I.F. : $e^{\int P(x)dx}$

because the left side is a derivative (product rule): $f\ g'$

$$\frac{d}{dx}\left(\underbrace{e^{\int P(x)dx}}_{f}\ \underbrace{y(x)}_{g}\right) = e^{\int P(x)dx} Q(x)\ .$$

$(fg)' = f'g + fg'$

$f' = e^{\int P(x)dx} \cdot P(x)$

So you can antidifferentiate both sides with respect to x :

antidiff

$$e^{\int P(x)dx} y = \int e^{\int P(x)dx} Q(x)dx \quad + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$. Notice, if you look carefully at this formula for the solution, that if $P(x)$, $Q(x)$ are defined and continuous on any interval I, then the resulting formula for $y(x)$ can be used to find a solution to any IVP with initial point in that interval, defined on the entire interval. This is in contrast to what can happen with separable differential equations.

so $y = \dfrac{1}{e^{\int P(x)dx}}\left[\qquad\qquad\right]$

Exercise 2: Solve the differential equation

$$y'(x) = -2\,y + 4\,x^2\,,$$

and compare your solutions to the dfield plot below.

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> with(DEtools) :    # load differential equations library
> deqtn2 := y'(x) = -2·y(x) + 4·x^2 :  #notice you must use · for multiplication in Maple,
                                        # and write y(x) rather than y.
  dsolve(deqtn2, y(x)); # Maple check
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$$\boxed{y(x) = 2\,x^2 - 2\,x + 1 + e^{-2\,x}\,\_C1} \qquad (7)$$

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>
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(1) $y'(x) + 2y(x) = 4x^2$

$y' + 2y = 4x^2$

(2) $P(x) = 2$; $\int P(x)dx = \int 2\,dx = 2x$

I.F. $e^{\int P(x)dx} = e^{2x}$

(3) $e^{2x}[y' + 2y] = e^{2x}\cdot 4x^2$

$\underbrace{\frac{d}{dx}\left[e^{2x}y\right]}_{f'g+fg'} = 4x^2e^{2x}$

$\boxed{2e^{2x}y} + e^{2x}y'$

(4) $e^{2x}y = \int \overbrace{4x^2}^{u}\,\overbrace{e^{2x}\,dx}^{dv}$ $du = 8x\,dx$ $v = \frac{e^{2x}}{2}$

$(= e^{2x}(2x^2 - 2x + 1) + C)$

$uv - \int v\,du$

$= 2x^2e^{2x} - \int \underbrace{4x}_{u}\,\underbrace{e^{2x}dx}_{dv}$ $\cdot\, du = 4dx \cdot v = \frac{e^{2x}}{2}$

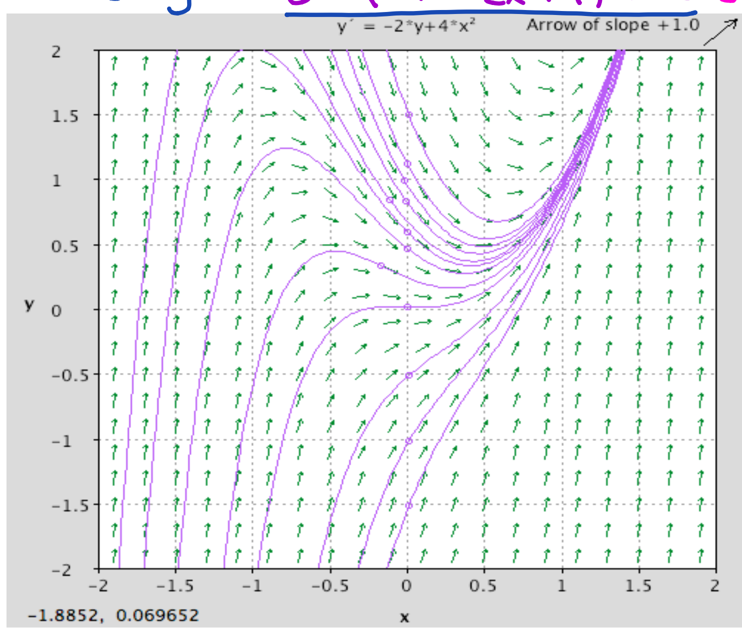
$= 2x^2e^{2x} - \left[2xe^{2x} - \underbrace{\int 2e^{2x}dx}_{e^{2x}}\right]$

$= 2x^2e^{2x} - 2xe^{2x} + e^{2x}$

$e^{2x}y = e^{2x}(2x^2 - 2x + 1) + C$

(5) $\div$ I.F.

$$\boxed{y(x) = 2x^2 - 2x + 1 + Ce^{-2x}}$$



Exercise 3: Find all solutions to the linear (and also separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2+1} = \frac{6x}{x^2+1} - \frac{3x}{x^2+1}\,y$$

Hint: as you can verify below, the general solution is $y(x) = 2 + C\left(x^2+1\right)^{-\frac{3}{2}}$.

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> with(DEtools):
  dsolve(y'(x) = (6·x − 3·x·y(x))/(x^2 + 1), y(x));  #Maple check
```

$$y(x) = 2 + \frac{\_C1}{\left(x^2+1\right)^{3/2}} \qquad \textbf{(8)}$$

$$y' + \underbrace{\frac{3x}{x^2+1}}_{P(x)}\,y = \frac{6x}{x^2+1}$$

(5) $\div$ I.F.

$$\boxed{y = 2 + C\left(x^2+1\right)^{-3/2}}$$

(1) $\int P(x)\,dx = \int \frac{3x}{x^2+1}\,dx = \int \frac{3}{2}\frac{du}{u} = \frac{3}{2}\ln|u| \cancel{+C}$

$= \frac{3}{2}\ln(x^2+1)$

$u = x^2+1$

$du = 2x\,dx$

$\frac{3}{2}du = 3x\,dx$

(2) I.F. $e^{\int P(x)dx} = e^{\frac{3}{2}\ln(x^2+1)} = \left[e^{\ln(x^2+1)}\right]^{\frac{3}{2}}$

$= (x^2+1)^{3/2}$

(3) $\underset{f}{(x^2+1)^{3/2}}\left[\underset{g'}{y'} + \frac{3x}{x^2+1}\overset{g}{y}\right] = (x^2+1)^{3/2}\cdot\frac{6x}{x^2+1}$ (f')

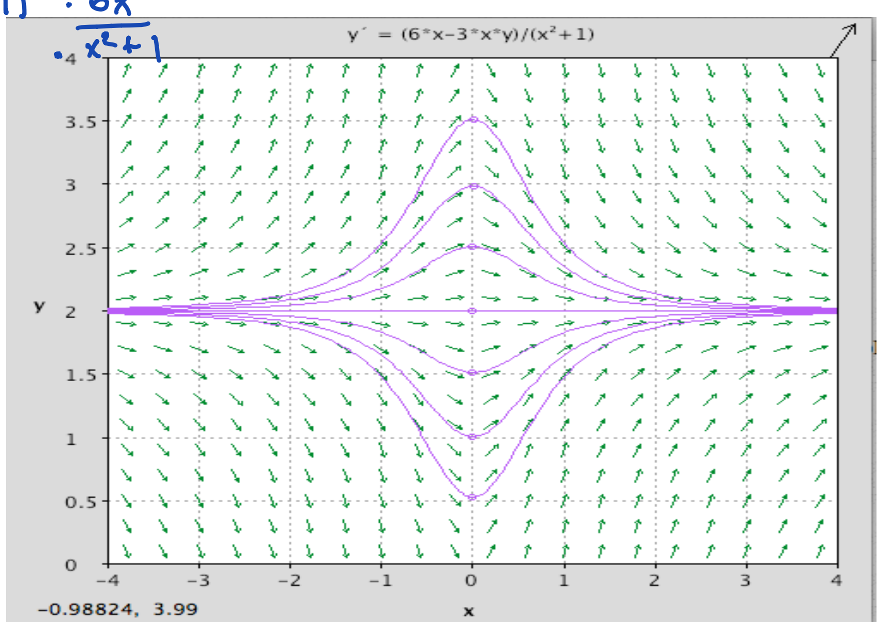
(4) $\frac{d}{dx}\left[\underset{f}{(x^2+1)^{3/2}}\,\underset{g}{y(x)}\right] = 6x\,(x^2+1)^{1/2}$

$f' = \frac{3}{\cancel{2}}(x^2+1)^{1/2}\cdot\cancel{2}x$

$(x^2+1)^{3/2}\,y = \int 6x\,(x^2+1)^{1/2}\,dx$

$(x^2+1)^{3/2}\,y = 2(x^2+1)^{3/2} + C$

$\frac{d}{dx}2(x^2+1)^{3/2} = \frac{3}{\cancel{2}}(x^2+1)^{1/2}\cdot\cancel{2}x$ ($\cdot 2$)



$$\frac{dy}{dx} = \frac{3x}{x^2+1}(2-y)$$

$$* \quad \frac{dy}{y-2} = -\frac{3x}{x^2+1}\, dx$$

($y \neq 2 \rightarrow y(x) \equiv 2$ is soln
$y' = 0$, RHS $= 0$)

I'll fill in the rest!

filled in: integrate *

$$\int \frac{dy}{y-2} = \int \frac{-3x}{x^2+1}\, dx \leftarrow$$

$u = x^2+1$
$du = 2x\, dx$
$-\frac{3}{2} du = -3x\, dx$

$$\ln|y-2| = -\frac{3}{2}\ln(x^2+1) + C$$

($u = y-2$, $du = dy$)

$$\int \frac{-\frac{3}{2} du}{u} = -\frac{3}{2}\ln|u| = -\frac{3}{2}\ln(x^2+1)$$

exponentiate:

$$|y-2| = e^{-\frac{3}{2}\ln(x^2+1)} e^{C_1}$$

$$y-2 = C(x^2+1)^{-3/2} \qquad C = \pm e^{C_1}$$

$$\boxed{y = 2 + C(x^2+1)^{-3/2}}$$

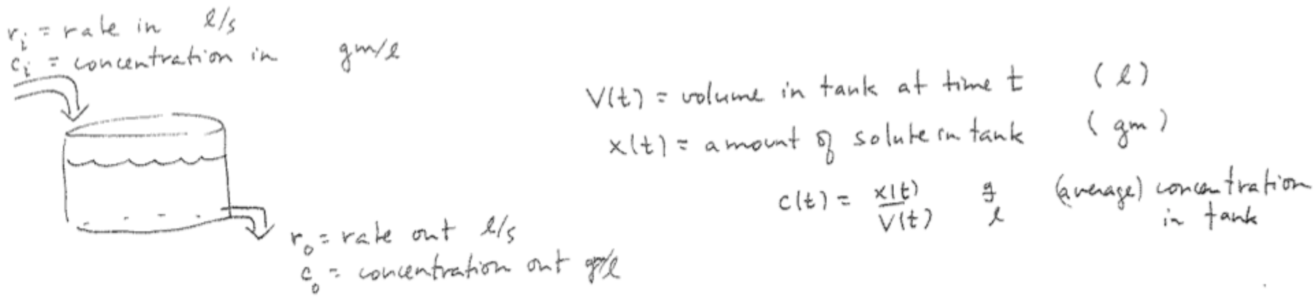
note $C = 0$ yields the singular soln $y = 2$

An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o, and with concentration c_0. We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t, given $V(0) = V_0,\ x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l}\ .$$

See the diagram below.



<u>Exercise 4:</u> Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:

a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a\,x = b$$
$$x(0) = x_0$$

where a, b are constants. This differential equation is separable and linear, and it is recommended that you become good at solving it. Notice that it includes the exponential growth/decay and Newton's law of cooling DE's as special cases.