- We'll use Wednesday's notes for a lot of the class

Math 2250-004
Fri Jan 13

- I'll post a copy of the lab in our HW page
- our CANVAS page exists
- next week's notes should be posted by 3:00 today. Print these yourself for Tuesday class

1.3-1.4: slope fields; existence and uniqueness for solutions to IVPs; examples we can check with separation of variables.

Exercise 1: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 \, . \quad = 1\cdot(1+y^2) \to \phi(y)$$

not §1.2 problem! $y'(x) = f(x)$, $y(x) = \int f(x)\,dx$

a) Use separation of variables to find solutions to this DE...the "magic" algorithm that we talked about at the start of the week, but didn't explain the reasoning for. It is de-mystified on the next page of today's notes.
b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a? (You can sketch by hand, I'll use "dfield" on my browser.)
c) Explain why each IVP has a solution, but this solution does not exist for all x.

You can download the java applet "dfield" from the URL
http://math.rice.edu/~dfield/dfpp.html
(You also have to download a toolkit, following the directions there.)

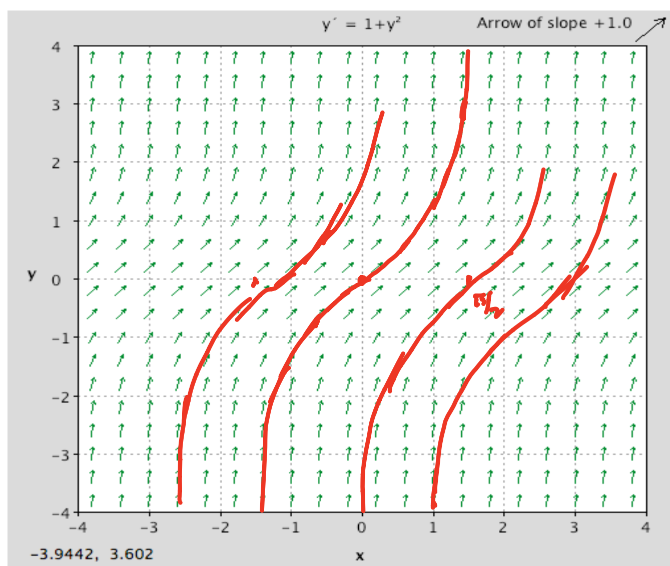
shorthand for $y'(x) = 1 + y(x)^2$

BAD: $\frac{dy}{dx} = 1 + y^2$

$\int \frac{dy}{dx}\,dx = \int 1 + y^2\,dx$ ($y(x)$)

$y = x + \,??$

don't know $y(x)$
don't know $y(x)^2$
so you can't compute $\int y(x)^2\,dx$



RIGHT WAY:

$$\frac{1}{1+y^2}\frac{dy}{dx} = 1$$

$$\int \frac{dy}{1+y^2} = \int dx \quad !$$

magic

$\arctan(y) = x + C$

$\tan(\arctan(y)) = \tan(x+C)$

$y = \tan(x+C)$

e.g. $C = 0$, $y = \tan x$ — vert. asym @ $x = \pm\frac{\pi}{2}$

solve IVP for $\begin{cases} y' = 1+y^2 \\ y(0) = 0 \end{cases}$

$y = \tan(x+C)$ related to $y = \tan(x)$ shifted horizontally by $(-C)$

$$\frac{dy}{dx} = x + xy = x(1+y)$$

$f(x)$ $\phi(y)$

1.4 Separable DE's: Important applications, as well as a lot of the examples we study in slope field discussions of section 1.3 are separable DE's. So let's discuss precisely what they are, and why the separation of variables algorithm works.

Definition: A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) .$$

if $y=c$ does make $\phi(c)=0$ then $y(x)\equiv c$ solves the DE!

check! in this case $y'(x)=0$ LHS

& RHS $= f(x)\phi(c) = f(x)\cdot 0 = 0$

It's more convenient to rewrite this DE as

$$\boxed{\frac{1}{\phi(y)}\frac{dy}{dx} = f(x),} \quad \boxed{\text{(as long as } \phi(y) \neq 0)}.$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$\boxed{g(y)\frac{dy}{dx} = f(x) .}$$

shorthand for

Solution (math justified): The left side of the modified differential equation is short for $g(y(x))\dfrac{dy}{dx}$. And if $G(y)$ is any antiderivative of $g(y)$, then we can rewrite this as

$G'(y) = g(y)$

$$G'(y(x))y'(x)$$

which by the chain rule (read backwards) is nothing more than

$$\frac{d}{dx}G(y(x)).$$

$$\int g(y(x))\, y'(x)\, dx = \int f(x)\, dx$$

$$\int \underbrace{G'(y(x))\, y'(x)}\, dx = \int f(x)dx$$

$$\int \frac{d}{dx} G(y(x))\, dx = \int f(x)dx$$

$$G(y(x)) + C_1 = F(x) + C_2$$

And the solutions to

$$\frac{d}{dx}G(y(x)) = f(x)$$

are

$$\boxed{G(y(x)) = \int f(x)\, dx = F(x) + C.}$$

where $F(x)$ is any antiderivative of $f(x)$. Thus solutions $y(x)$ to the original differential equation satisify

i.e $\boxed{G(y) = F(x) + C .}$

This expresses solutions $y(x)$ implicitly as functions of x. You may be able to use algebra to solve this equation explicitly for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

Solution (differential magic): Treat $\dfrac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} .$$

cross multiply

$$g(y)dy = f(x)dx .$$

Antidifferentiate each side with respect to its variable (?!)

magic!

$$\int g(y)dy = \int f(x)dx \text{, i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow \boxed{G(y) = F(x) + C .} \quad \text{Agrees!}$$

This is the same differential magic that you used for the "method of substitution" in antidifferentiation, which was essentially the "chain rule in reverse" for integration techniques.

Exercise 2a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$\boxed{y(0) = 0}$$

2b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

2c) Sketch some of these singular solutions onto the slope field below.

2a) $\frac{dy}{dx} = y^{2/3}$

$\int \frac{1}{y^{2/3}}\,dy = \int 1 \cdot dx$ ($y \neq 0$) → if $y(x) \equiv 0$ that would be a soltn, because $y'(x) = 0$ LHS, $y^{2/3}(x) = 0$ RHS

$\int y^{-2/3}\,dy = \int dx$

$3y^{1/3} = x + C$ implicit soltn

$y^{1/3} = \frac{1}{3}(x+C)$

$\boxed{y = \frac{1}{27}(x+C)^3}$ (as long as $x = -c$)

IVP: $y(0) = 0$

$0 = y(0) = \frac{1}{27}(0+c)^3 \Rightarrow C = 0$

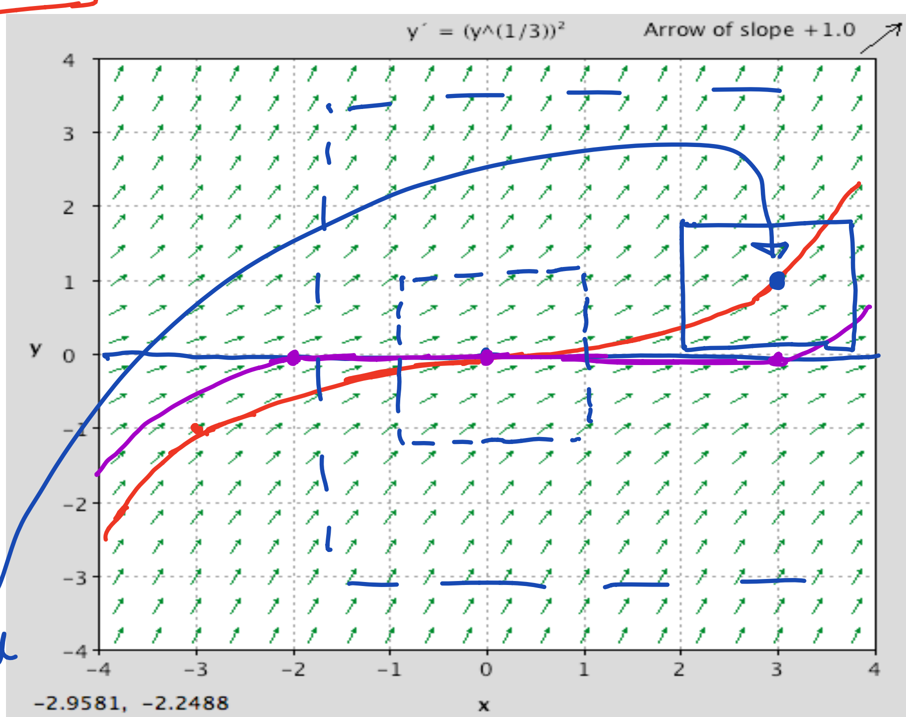
$\boxed{y(x) = \frac{1}{27}x^3}$ is a soltn:

$y'(x) = \frac{1}{9}x^2$

$y^{2/3} = \left(\frac{1}{27}x^3\right)^{2/3} = \frac{1}{9}\cdot x^2$

agree!

2b) $y(x) \equiv 0$ also solves IVP



$\begin{cases} y' = y^{2/3} \\ y(3) = 1 \end{cases}$

this IVP would have unique soltn as long graph stayed in rectangle which avoided x-axis ($x > 0$)

2c) e.g.

$$y(x) = \begin{cases} \frac{1}{27}(x+2)^3 & x \leq -2 \\ 0, & -2 \leq x \leq 3 \\ \frac{1}{27}(x-3)^3 & x \geq 3 \end{cases}$$

∞'ly many soltns to this IVP!!

how to compute $\frac{\partial}{\partial y} f(x,y)$ treat "x" as const. and take deriv with respect to y

$y' = y^{2/3}$

slope fcn $f(x,y) = y^{2/3} = (\sqrt[3]{y})^2$ is continuous on all of $\mathbb{R}^2$

• part 1 of $\exists!$ theorem, IVP $\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases}$ ($R = \mathbb{R}^2$) has at least one soltn.

• part 2: $\frac{\partial}{\partial y} y^{2/3} = \frac{2}{3}y^{-1/3}$ continuous except at $y = 0$ i.e. along x-axis

theorem guaranteeing uniqueness fails for OUR IVP

Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)
<u>Existence - uniqueness theorem for the initial value problem</u>
Consider the IVP

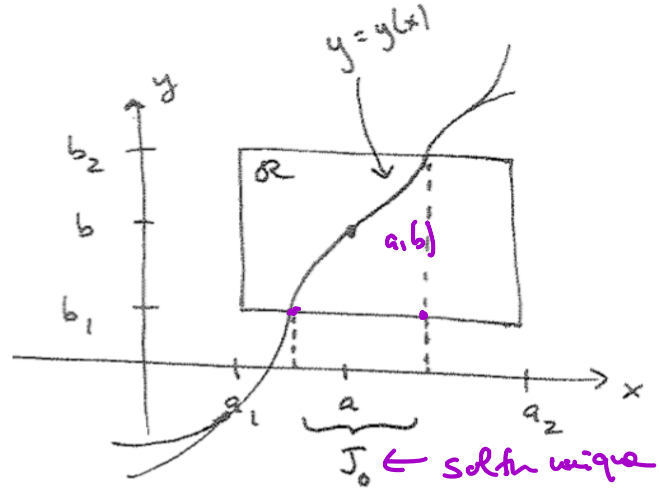
$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(a) = b \end{cases}$$

• Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : \underbrace{a_1 \le x \le a_2}_{a_1 < a < a_2}, \underbrace{b_1 \le y \le b_2}_{b_1 < b < b_2}$ in the x-y plane.

• <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$. "$\exists$" there exists.

• <u>Uniqueness:</u> If the partial derivative function $\dfrac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 3</u>: Discuss how the existence-uniqueness theorem is consistent with our work in Wednesday's Exercises 1-2, and in today's Exercises 1-2 where we were able to find explicit solution formulas because the differential equations were actually separable.

• You need to print out your own notes from now on. ☺ free in Math Tutoring Center

Math 2250-004 : Week 2, Jan 17-20; material from sections 1.3, 1.4, 1.5, EP 3.7
Tues Jan 17

• HW due tomorrow @ start of class (Labs due Thursday)
• Office hours 4:30-6:00 pm today LCB 218

We will mostly use last Friday's notes. Our goals today are

- (1) understand what makes a first order differential equation separable.
- (2) understand the algorithm based on differentials that solves separable differential equations: why it works, and how it sometimes misses "singular solutions"
- (3) understand and apply the existence-uniqueness theorem for first order DE initial value problems. §1.3

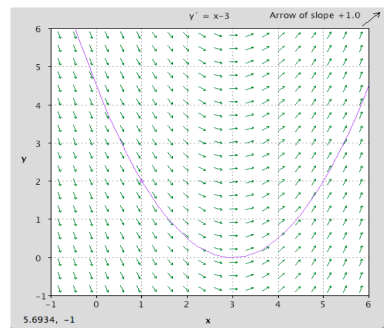
When discussing the existence-uniqueness theorem at the end of Friday's notes today, we'll refer to examples from Wednesday's notes that we discussed on Friday. Those were:

Exercise 1 (Wednesday notes, discussed Friday):

$$\frac{dy}{dx} = \boxed{x-3}$$
$$y(1) = 2\,.$$

slope fcn $f(x,y) = x-3$ is cont. on all of $\mathbb{R}^2$
• soltns to all IVP's exist
$\frac{\partial f}{\partial y} = 0$ is cont on $\mathbb{R}^2$
• soltns to IVP are unique

We found the solution $y(x) = \frac{x^2}{2} - 3x + \frac{9}{2} = \frac{(x-3)^2}{2}$. Is this consistent with the existence-uniqueness theorem?



Exercise 2: (Wednesday notes, discussed Friday)

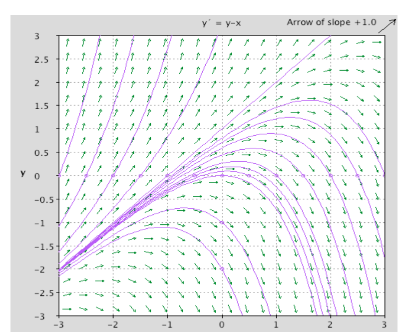
$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

From a family of solutions that was given to us, we found a solution

$$\boxed{y(x) = x + 1 - e^x\,.}$$

Is this the only possible solution? Hint: use the existence-uniqueness theorem.

slope fcn $f(x,y) = y-x$ cont on $\mathbb{R}^2$
$\Rightarrow$ IVP has solutions
$\frac{\partial f}{\partial y} = 1$ also cont.
so IVP soltns are unique.



Exercise 1 (today): Here's another example of using a separable DE to illustrate the existence-uniqueness theorem.

a) Does each IVP

$$y' = x^2 y^2$$
$$y(x_0) = y_0$$

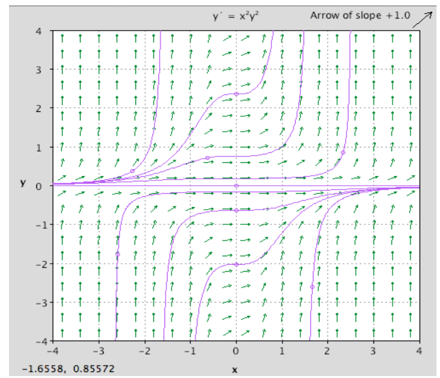
have a unique solution?

$f(x,y) = x^2y^2$ cont on $\mathbb{R}^2$

$\frac{\partial f}{\partial y} = x^2 2y$ cont on $\mathbb{R}^2$

So soltns to IVP's exist & are unique!

b) Find all solutions to this differential equation.



Maple check (notice it misses the singular solution):

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> with(DEtools) :
  dsolve(y'(x) = x^2·y(x)^2, y(x));
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$$y(x) = \frac{3}{-x^3 + 3\_C1} \qquad \textbf{(1)}$$

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>
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Exercise 2: Do the initial value problems below always have unique solutions? Can you find them? (Notice these are NOT separable differential equations.) Can Maple find formulas for the solution functions?

a)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

b)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$