- We'll use Wednesday's notes for a lot of the class

Math 2250-004
Fri Jan 13

- I'll post a copy of the lab in our HW page
- our CANVAS page exists
- next week's notes should be posted by 3:00 today. Print these yourself for Tuesday class

1.3-1.4: slope fields; existence and uniqueness for solutions to IVPs; examples we can check with separation of variables.

Exercise 1: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 \; .$$

$= 1 \cdot (1+y^2)$ $\quad \phi(y)$

not §1.2 problem! $y'(x) = f(x)$, $y(x) = \int f(x)\,dx$

a) Use separation of variables to find solutions to this DE...the "magic" algorithm that we talked about at the start of the week, but didn't explain the reasoning for. It is de-mystified on the next page of today's notes.
b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a? (You can sketch by hand, I'll use "dfield" on my browser.)
c) Explain why each IVP has a solution, but this solution does not exist for all x.

You can download the java applet "dfield" from the URL
http://math.rice.edu/~dfield/dfpp.html
(You also have to download a toolkit, following the directions there.)

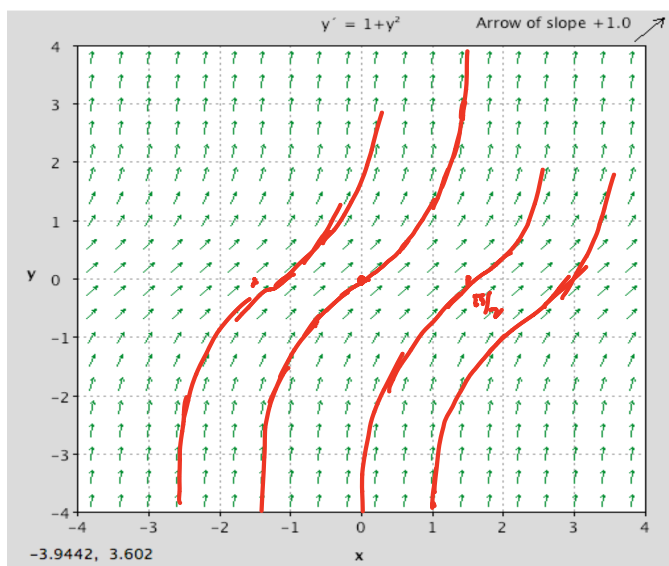
shorthand for $y'(x) = 1 + y(x)^2$

BAD: $\frac{dy}{dx} = 1 + y^2$

$\int \frac{dy}{dx}\,dx = \int 1 + y^2\,dx$ $\quad y(x)$

$y = x + ??$

don't know $y(x)$
don't know $y(x)^2$
so you can't compute $\int y(x)^2\,dx$



RIGHT WAY:

$\frac{1}{1+y^2}\frac{dy}{dx} = 1$

$\int \frac{dy}{1+y^2} = \int dx$!

magic

$\arctan(y) = x + C$

$\tan(\arctan(y)) = \tan(x+C)$

$y = \tan(x+C)$

e.g. $C=0$, $y = \tan x$ — vert. asym @ $x = \pm\pi/2$

solve IVP for $\begin{cases} y' = 1+y^2 \\ y(0) = 0 \end{cases}$

$y = \tan(x+C)$ related to $y = \tan(x)$ shifted horizontally by $(-C)$

$$\frac{dy}{dx} = x + xy = \underset{f(x)}{x}\,\underset{\phi(y)}{(1+y)}$$

1.4 Separable DE's: Important applications, as well as a lot of the examples we study in slope field discussions of section 1.3 are separable DE's. So let's discuss precisely what they are, and why the separation of variables algorithm works.

Definition: A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) \, .$$

if $y=c$ does make $\phi(c)=0$ then $y(x)\equiv c$ solves the DE!

check! in this case $y'(x)=0$ LHS

& RHS $= f(x)\phi(c) = f(x)\cdot 0 = 0$

It's more convenient to rewrite this DE as

$$\boxed{\frac{1}{\phi(y)}\frac{dy}{dx} = f(x),} \quad \boxed{\text{(as long as } \phi(y) \neq 0)}.$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$\boxed{g(y)\frac{dy}{dx} = f(x) \, .}$$

shorthand for

Solution (math justified): The left side of the modified differential equation is short for $g(y(x))\dfrac{dy}{dx}$. And if $G(y)$ is any antiderivative of $g(y)$, then we can rewrite this as

$G'(y) = g(y)$

$$G'(y(x))y'(x)$$

which by the chain rule (read backwards) is nothing more than

$$\frac{d}{dx}G(y(x)).$$

$$\int g(y(x))\, y'(x)\, dx = \int f(x)\, dx$$

$$\int \underbrace{G'(y(x))\, y'(x)}\, dx = \int f(x)dx$$

$$\int \frac{d}{dx}G(y(x))\, dx = \int f(x)dx$$

$$G(y(x)) + C_1 = F(x) + C_2$$

And the solutions to

$$\frac{d}{dx}G(y(x)) = f(x)$$

are

$$\boxed{G(y(x)) = \int f(x)\, dx = F(x) + C.}$$

where $F(x)$ is any antiderivative of $f(x)$. Thus solutions $y(x)$ to the original differential equation satisify

$$\boxed{G(y) = F(x) + C \, .}$$

This expresses solutions $y(x)$ i.e implicitly as functions of x. You may be able to use algebra to solve this equation explicitly for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

Solution (differential magic): Treat $\dfrac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \, .$$

cross multiply

$$g(y)dy = f(x)dx \, .$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx \text{, i.e.}$$

magic!

$$G(y) + C_1 = F(x) + C_2 \Rightarrow \boxed{G(y) = F(x) + C} \quad \text{Agrees!}$$

This is the same differential magic that you used for the "method of substitution" in antidifferentiation, which was essentially the "chain rule in reverse" for integration techniques.

Exercise 2a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$
$$\boxed{y(0) = 0}$$

2b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

2c) Sketch some of these singular solutions onto the slope field below.

2a) $\frac{dy}{dx} = y^{2/3}$

$\int \frac{1}{y^{2/3}}\,dy = \int 1\cdot dx$ $(y \neq 0) \to$ if $y(x) \equiv 0$ that would be a soln, because $y'(x) = 0$ LHS, $y^{2/3}(x) = 0$ RHS

$\int y^{-2/3}\,dy = \int dx$

$3y^{1/3} = x + C$ implicit soln

$y^{1/3} = \frac{1}{3}(x+C)$

$\boxed{y = \frac{1}{27}(x+C)^3}$ (as long as $x = -c$)

IVP: $y(0) = 0$

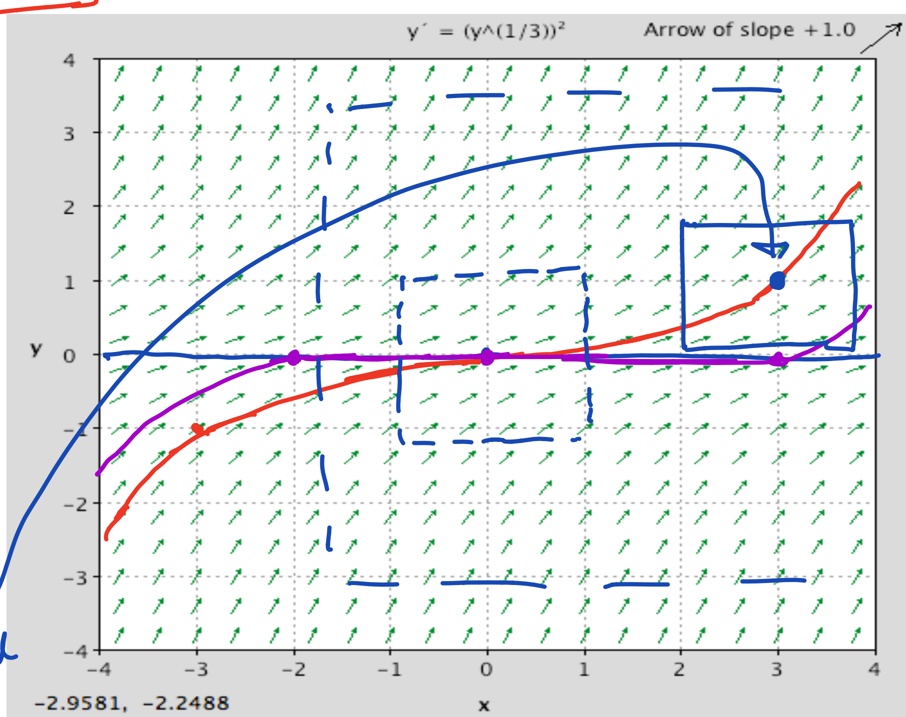
$0 = y(0) = \frac{1}{27}(0+c)^3 \Rightarrow c = 0$

$\boxed{y(x) = \frac{1}{27}x^3}$ is a soln:

$y'(x) = \frac{1}{9}x^2$

$y^{2/3} = \left(\frac{1}{27}x^3\right)^{2/3} = \frac{1}{9}\cdot x^2$

agree!



2b) $y(x) \equiv 0$ also solves IVP

$\begin{cases} y' = y^{2/3} \\ y(3) = 1 \end{cases}$

this IVP would have unique soln as long graph stayed in rectangle which avoided x-axis ($x > 0$)

2c) e.g.

$$y(x) = \begin{cases} \frac{1}{27}(x+2)^3 & x \le -2 \\ 0, & -2 \le x \le 3 \\ \frac{1}{27}(x-3)^3 & x \ge 3 \end{cases}$$

∞'ly many solns to this IVP!!

how to compute $\frac{\partial}{\partial y} f(x,y)$

treat "x" as const. and take deriv with respect to y

$y' = y^{2/3}$ slope fn $f(x,y) = y^{2/3} = (\sqrt[3]{y})^2$ is continuous on all of $\mathbb{R}^2$

• part 1 of $\exists!$ theorem, IVP $\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases}$ ($R = \mathbb{R}^2$) has at least one soln.

• part 2: $\frac{\partial}{\partial y} y^{2/3} = \frac{2}{3}y^{-1/3}$ continuous except at $y = 0$ i.e. along x-axis

theorem guaranteeing uniqueness fails for OUR IVP

Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)

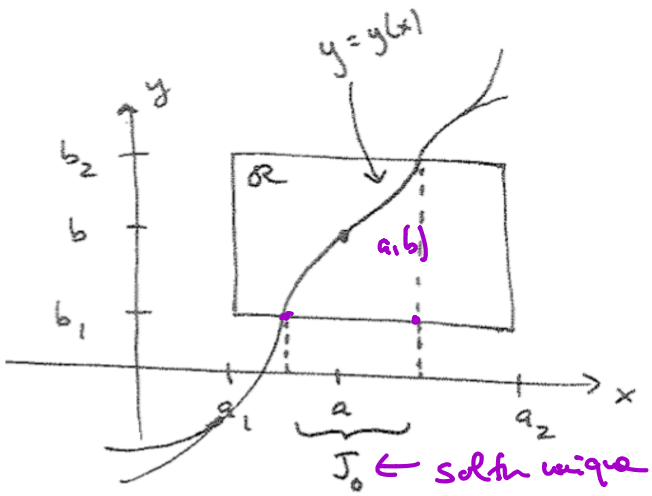
Existence - uniqueness theorem for the initial value problem

Consider the IVP

$$\begin{cases} \dfrac{dy}{dx} = f(x, y) \\ y(a) = b \end{cases}$$

- Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R}$: $\underbrace{a_1 \leq x \leq a_2}_{a_1 < a < a_2}$, $\underbrace{b_1 \leq y \leq b_2}_{b_1 < b < b_2}$ in the x-y plane.
- Existence: If $f(x, y)$ is continuous in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$. "$\exists$" there exists.
- Uniqueness: If the partial derivative function $\dfrac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



Exercise 3: Discuss how the existence-uniqueness theorem is consistent with our work in Wednesday's Exercises 1-2, and in today's Exercises 1-2 where we were able to find explicit solution formulas because the differential equations were actually separable.

• You need to print out your own notes from now on. ☺ free in Math Tutoring Center

Math 2250-004 : Week 2, Jan 17-20; material from sections 1.3, 1.4, 1.5, EP 3.7
Tues Jan 17
We will mostly use last Friday's notes. Our goals today are

• HW due tomorrow @ start of class (Labs due Thursday)
• office hours 4:30-6:00 pm today LCB 218

- (1) understand what makes a first order differential equation separable.
- (2) understand the algorithm based on differentials that solves separable differential equations: why it works, and how it sometimes misses "singular solutions"
- (3) understand and apply the existence-uniqueness theorem for first order DE initial value problems. §1.3

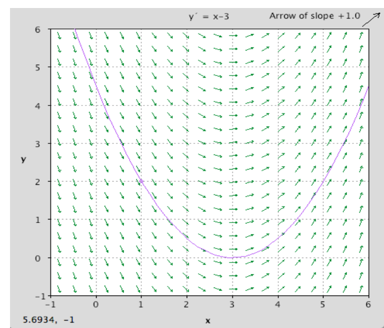
When discussing the existence-uniqueness theorem at the end of Friday's notes today, we'll refer to examples from Wednesday's notes that we discussed on Friday. Those were:

Exercise 1 (Wednesday notes, discussed Friday):

$$\frac{dy}{dx} = \boxed{x - 3}$$
$$y(1) = 2 .$$

slope fcn $f(x,y) = x-3$ is cont. on all of $\mathbb{R}^2$
• solfns to all IVP's exist
$\frac{\partial f}{\partial y} = 0$ is cont on $\mathbb{R}^2$
• solfns to IVP are unique

We found the solution $y(x) = \frac{x^2}{2} - 3x + \frac{9}{2} = \frac{(x-3)^2}{2}$. Is this consistent with the existence-uniqueness theorem?



Exercise 2: (Wednesday notes, discussed Friday)

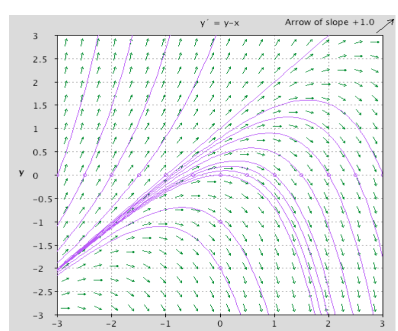
$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

From a family of solutions that was given to us, we found a solution

$$\boxed{y(x) = x + 1 - e^x .}$$

Is this the only possible solution? Hint: use the existence-uniqueness theorem.

slope fcn $f(x,y) = y - x$ cont on $\mathbb{R}^2$
$\Rightarrow$ IVP has solutions
$\frac{\partial f}{\partial y} = 1$ also cont.
so IVP solfns are unique.



Exercise 1 (today): Here's another example of using a separable DE to illustrate the existence-uniqueness theorem.

a) Does each IVP

$$y' = x^2 y^2$$
$$y(x_0) = y_0$$

have a unique solution?

$\left\{ \begin{array}{ll} f(x,y) = x^2y^2 & \text{cont on } \mathbb{R}^2 \\ \frac{\partial f}{\partial y} = x^2 2y & \text{cont on } \mathbb{R}^2 \end{array} \right.$

So sol'ns to IVP's exist & are unique!

b) Find all solutions to this differential equation.

And illustrate dfield.

Wednesday starts here

$$\frac{dy}{dx} = x^2y^2$$

$$\int \frac{dy}{y^2} = \int x^2\,dx \quad \text{if } y \neq 0$$

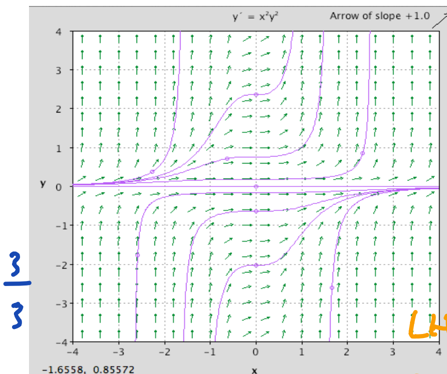
$$\frac{1}{y^2} = y^{-2}, \quad \int y^{-2}\,dy = \frac{y^{-1}}{-1} = \frac{x^3}{3} + C$$

$$-\frac{1}{y} = \frac{x^3}{3} + C$$

$$\Rightarrow -y = \frac{1}{\frac{x^3}{3} + C}$$

$$\Rightarrow y = \frac{-1}{\frac{x^3}{3} + C} \cdot \frac{3}{3}$$

$$\boxed{y = \frac{-3}{x^3 + 3C}} \text{ "=" } \frac{-3}{x^3 + C}$$



if $y = 0$, $y(x) = 0$ is also a sol'n - then LHS $y'(x) = 0$, RHS $x^2y^2 = 0$

Maple check (notice it misses the singular solution):

```
> with(DEtools) :
  dsolve(y'(x) = x^2·y(x)^2, y(x));
```

$$y(x) = \frac{3}{-x^3 + 3\_C1} \qquad (1)$$

$\left\{ \begin{array}{l} y' = x^2y^2 \\ y(0) = 0 \end{array} \right.$ won't work

soln: $y(x) = 0$

```
>
```

Exercise 2: Do the initial value problems below always have unique solutions? Can you find them? (Notice these are NOT separable differential equations.) Can Maple find formulas for the solution functions?

a)

$$y' = x^2 + y^2$$
$$y(x_0) = y_0$$

slope function $f(x,y) = x^2 + y^2$ is cont on $\mathbb{R}^2$ (entire x-y plane)

$\frac{\partial f}{\partial y} = 2y$ also cont on $\mathbb{R}^2$

"there exists" $\hookrightarrow \exists!$ sol'ns to each IVP.

"unique"

b)

$$y' = x^4 + y^4$$
$$y(x_0) = y_0$$

$\left. \begin{array}{l} f = x^4 + y^4 \\ \frac{\partial f}{\partial y} = 4y^3 \end{array} \right\}$ continuous on $\mathbb{R}^2$

In Maple, we saw that even though sol'ns exist, there might not be a formula for them

Math 2250-004
Wed Jan 18 Quiz at end of class!
1.4: separable DEs, examples and experiment.

- Lab 2 tomorrow
- bring laptops to lab
- Quiz today
- Office hours today 4:30-6:00 LCB 218
- little bit of Tuesday

For your section 1.4 hw this week I assigned a selection of separable DE's - some applications will be familiar with from last week, e.g. exponential growth/decay and Newton's Law of cooling. Below is an application that might be new to you, and that illustrates conservation of energy as a tool for modeling differential equations in physics.

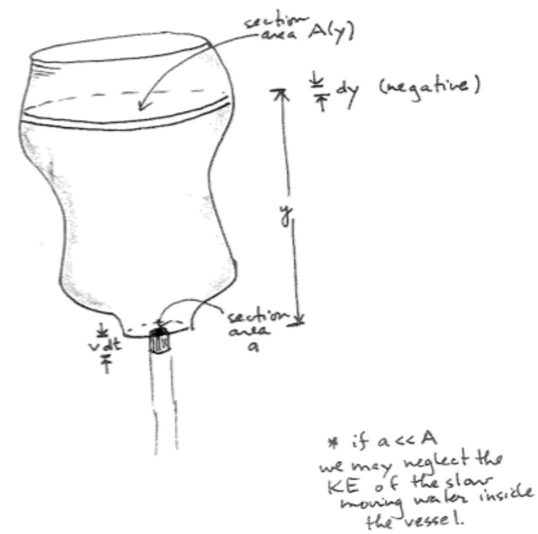
<u>Toricelli's Law</u>, for draining water tanks. Refer to the figure below.
<u>Exercise 1</u>:
a) Neglect friction, use conservation of energy, and assume the water still in the tank is moving with negligable velocity $(a << A)$. Equate the lost potential energy from the top in time dt to the gained kinetic energy in the water streaming out of the hole in the tank to deduce that the speed v with which the water exits the tank is given by

$$v = \sqrt{2 g y}$$

when the water depth above the hole is $y(t)$ (and g is accel of gravity).

b) Use part (a) to derive the separable DE for water depth

$$A(y)\frac{dy}{dt} = -k\sqrt{y} \quad (k = a\sqrt{2 g}) .$$



Experiment fun! I've brought a leaky nalgene canteen so we can test the Toricelli model. For a cylindrical tank of height h as below, the cross-sectional area $A(y)$ is a constant A, so the Toricelli DE and IVP becomes

$$\frac{dy}{dt} = -k\,y^{\frac{1}{2}}$$
$$y(0) = h$$

(different k).

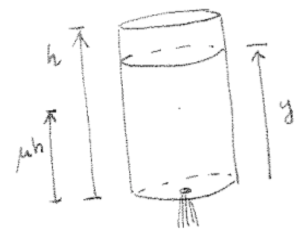
Exercise 2a) Solve the differential equation IVP, and IVP. Note that $y \geq 0$, and that $y = 0$ is a singular solution that separation of variables misses. We may choose our units of length so that $h = 1$ is the maximum water height in the tank. Show that in this case the solution to the IVP is given by

$$y(t) = \left(1 - \frac{k}{2}t\right)^2$$

(until the tank runs empty).

Exercise 2b: (We will use this calculation in our experiment) Setting the height $h = 1$ as in part 2a, let $T(\mu)$ be the time it takes the the water to go from height 1 (full) to height μ, where the fraction μ is between 0 and 1. Note, $T(1) = 0$ and $T(0)$ is the time it takes for the tank to empty completely. Show that $T(0)$ is related to $T(\mu)$ by

$$T(0)\left(1-\sqrt{\mu}\right) = T(\mu)\,, \quad \text{i.e. } T(0) = \frac{T(\mu)}{1-\sqrt{\mu}}.$$



Experiment! We'll time how long it takes to half-empty the canteen, and predict how long it will take to completely empty it when we rerun the experiment. Here are numbers I once got in my office, let's see how ours compare.

> $Digits := 5:$ *# that should be enough significant digits*

> $\dfrac{1}{1 - \text{sqrt}(.5)}$; *# the factor from above, when mu is 0.5*

$$3.4143 \tag{2}$$

> $Thalf := 35;$ *# seconds to half-empty canteen*
> $Tpredict := 3.4143 \cdot Thalf;$ *#prediction*

$$Thalf := 35$$
$$Tpredict := 119.50 \tag{3}$$

Remark: Maple can draw direction fields, although they're not as easy to create as in "dfield". On the other hand, Maple can do any undergraduate mathematics computation, including solving pretty much any differential equation that has a closed form solution. Let's see what the commands below produce. We'll get some error messages related to the existence-uniqueness theorem!

> *with*(*DEtools*) : # *this loads a library of DE commands*

> $deqtn1 := y'(t) = -\sqrt{y(t)}$; # *took k=1*
$ics1 := y(0) = 1$; # *took initial height=1*

$$deqtn1 := \mathrm{D}(y)(t) = -\sqrt{y(t)}$$

$$ics1 := y(0) = 1 \tag{4}$$

> *dsolve*({*deqtn1*, *ics1*}, *y*(*t*)); # *DE IVP sol!*

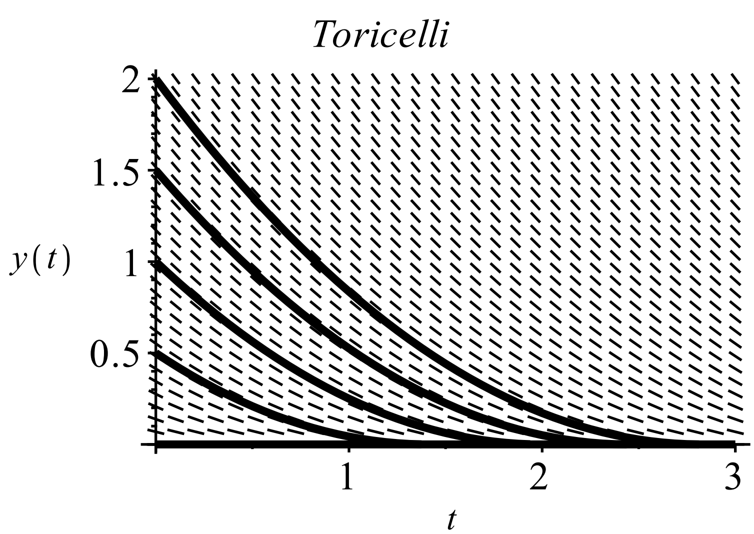
$$y(t) = \frac{1}{4}\,t^2 - t + 1 \tag{5}$$

> *factor*(%); #*how we wrote it*

$$y(t) = \frac{1}{4}\,(t-2)^2 \tag{6}$$

> *DEplot*(*deqtn1*, *y*(*t*), 0 ..3, {[*y*(0) = 0], [*y*(0) = 1.], [*y*(0) = 2.], [*y*(0) = 0.5], [*y*(0) = 1.5]}, *arrows* = *line*, *color* = *black*, *linecolor* = *black*, *dirgrid* = [30, 30], *stepsize* = .1, *title* = \`*Toricelli*\`);

```
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 1.4141924, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 1.9999776, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 2.4494671, probably a singularity
Warning, plot may be incomplete, the following errors(s) were issued:
   cannot evaluate the solution further right of 2.8283979, probably a singularity
```



>

· postpone experiment.
- so we can talk about 9.1.S.
· next week's notes are posted
· printing is free in Math tutoring Ctr. basement JWB, LCB

Math 2250-004
Fri Jan 20
1.5: linear DEs, and applications.

Section 1.5, linear differential equations: → because y', y appear to the 1^{st} power, possibly multiplied by a fcn of x

A first order linear DE for $y(x)$ is one that can be written as

$$y' + P(x)y = Q(x)$$

or if $x = x(t)$ $\quad x'(t) + P(t)x(t) = Q(t)$

Exercise 1: Classify the differential equations below as linear, separable, both, or neither. Justify your answers.

$x' + P(t)x = Q(t)$

$\left(\frac{dy}{dx} = f(x)g(y)\right)$ sep

a) $y'(x) = -2y + 4x^2$
b) $y'(x) = x - y^2 + 1$
c) $y'(x) = x^2 - x^2 y + 1$
d) $y'(x) = \dfrac{6x - 3xy}{x^2 + 1}$
e) $y'(x) = x^2 + y^2$
f) $y'(x) = x^2 e^{x^3}$.

9.1.2

	linear	separable
a)	✓	no
b) (bad y^2)	no	no
c)	✓	no
d)	✓	✓
e)	no	no
f)	✓	✓

a) $y' + \underbrace{2}_{P(x)}y = \underbrace{4x^2}_{Q(x)}$

b) can't have a y^2 term, must be y^1 for linear

c) $y' + x^2 y = x^2 + 1$

d) $y' + \underbrace{\frac{3x}{x^2+1}}_{P(x)} y = \frac{6x}{x^2+1}$ linear

$\frac{dy}{dx} = \frac{3x}{x^2+1}(2-y)$ sep.

or $\frac{x}{x^2+1}(6-3y)$

e)

f) $y' + 0y = x^2e^{x^3}$ $(P(x) \equiv 0)$ $\quad \frac{dy}{dx} = x^2e^{x^3}\cdot 1$

Algorithm for solving linear DEs is a method to use the differentiation product rule backwards:

$$y' + P(x)y = Q(x)$$

Let $\int P(x)dx$ be any antiderivative of P. Multiply both sides of the DE by its exponential to yield an equivalent DE:

integrating factor
I.F.

$$\underbrace{e^{\int P(x)dx}}_{f'}\underbrace{(y' + P(x)y)}_{g} = e^{\int P(x)dx}Q(x)$$

. equivalent to

I.F. : $e^{\int P(x)dx}$

because the left side is a derivative (product rule):

$$\frac{d}{dx}\left(\underbrace{e^{\int P(x)dx}}_{f}\underbrace{y(x)}_{g}\right) = e^{\int P(x)dx}Q(x).$$

$(fg)' = f'g + fg'$

$f' = e^{\int P(x)dx}\cdot P(x)$

So you can antidifferentiate both sides with respect to x:

antidiff

$$e^{\int P(x)dx}y = \int e^{\int P(x)dx}Q(x)dx + C.$$

Dividing by the positive function $e^{\int P(x)dx}$ yields a formula for $y(x)$. Notice, if you look carefully at this formula for the solution, that if $P(x)$, $Q(x)$ are defined and continuous on any interval I, then the resulting formula for $y(x)$ can be used to find a solution to any IVP with initial point in that interval, defined on the entire interval. This is in contrast to what can happen with separable differential equations.

so $y = \frac{1}{e^{\int P(x)dx}}\left[\quad\right]$

Exercise 2: Solve the differential equation

$$y'(x) = -2\,y + 4\,x^2\,,$$

and compare your solutions to the dfield plot below.

```
> with(DEtools) :    # load differential equations library
> deqtn2 := y'(x) = -2·y(x) + 4·x^2 :  #notice you must use · for multiplication in Maple,
                                        # and write y(x) rather than y.
  dsolve(deqtn2, y(x)); # Maple check
```

$$\boxed{y(x) = 2\,x^2 - 2\,x + 1 + e^{-2\,x}\,\_C1} \qquad (7)$$

```
>
```

① $y'(x) + 2y(x) = 4x^2$

$y' + 2y = 4x^2$

② $P(x) = 2$; $\int P(x)dx = \int 2\,dx = 2x$

I.F. $e^{\int P(x)dx} = e^{2x}$

③ $e^{2x}[y' + 2y] = e^{2x}\cdot 4x^2$

$\underbrace{\frac{d}{dx}\left[e^{2x}y\right]}_{f'g+fg'} = 4x^2e^{2x}$

$\boxed{2e^{2x}y} + e^{2x}y'$

④ $e^{2x}y = \int \underbrace{4x^2}_{u}\,\underbrace{e^{2x}\,dx}_{dv}$ $\qquad du = 8x\,dx \quad v = \frac{e^{2x}}{2}$

$(= e^{2x}(2x^2-2x+1) + C)$

$uv - \int v\,du$

$= 2x^2e^{2x} - \int \underbrace{4x}_{u}\,\underbrace{e^{2x}\,dx}_{dv}$ $\qquad du = 4dx \cdot v = \frac{e^{2x}}{2}$

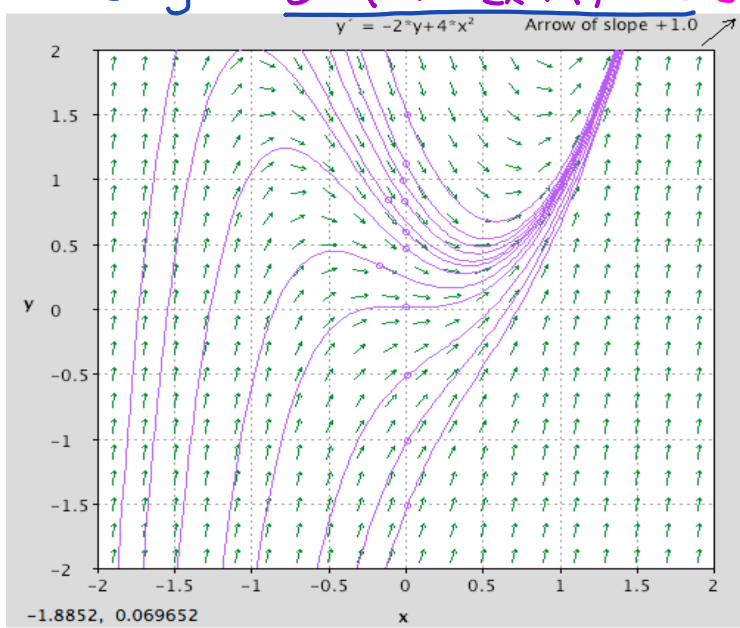
$= 2x^2e^{2x} - \left[2xe^{2x} - \underbrace{\int 2e^{2x}dx}_{e^{2x}}\right]$

$= 2x^2e^{2x} - 2xe^{2x} + e^{2x}$

$e^{2x}y = e^{2x}(2x^2-2x+1) + C$

⑤ $\div$ I.F.

$$\boxed{y(x) = 2x^2 - 2x + 1 + Ce^{-2x}}$$



Exercise 3: Find all solutions to the linear (and also separable) DE

$$y'(x) = \frac{6x - 3xy}{x^2+1} = \frac{6x}{x^2+1} - \frac{3x}{x^2+1}\,y$$

Hint: as you can verify below, the general solution is $y(x) = 2 + C\,(x^2+1)^{-\frac{3}{2}}$.

> *with*(*DEtools*) :

dsolve$\left(y'(x) = \frac{(6\cdot x - 3\cdot x\cdot y(x))}{x^2+1}, y(x)\right)$; *#Maple check*

$$y(x) = 2 + \frac{\_C1}{(x^2+1)^{3/2}} \qquad \textbf{(8)}$$

$$y' + \underbrace{\frac{3x}{x^2+1}}_{P(x)}\,y = \frac{6x}{x^2+1}$$

① $\int P(x)\,dx = \int \frac{3x}{x^2+1}\,dx = \int \frac{3}{2}\frac{du}{u} = \frac{3}{2}\ln|u| \cancel{+C}$

$u = x^2+1$

$du = 2x\,dx$

$\frac{3}{2}du = 3x\,dx$

$= \frac{3}{2}\ln(x^2+1)$

② I.F. $e^{\int P(x)dx} = e^{\frac{3}{2}\ln(x^2+1)} = \left[e^{\ln(x^2+1)}\right]^{\frac{3}{2}}$

$= (x^2+1)^{3/2}$

③ $\underbrace{(x^2+1)^{3/2}}_{f}\left[\underbrace{y' + \frac{3x}{x^2+1}y}_{g'}\right] = (x^2+1)^{3/2}\cdot\frac{6x}{x^2+1}$

f', g

④ $\frac{d}{dx}\left[\underbrace{(x^2+1)^{3/2}}_{f}\underbrace{y(x)}_{g}\right] = 6x\,(x^2+1)^{1/2}$

$f' = \frac{3}{\cancel{2}}(x^2+1)^{1/2}\cdot\cancel{2}x$

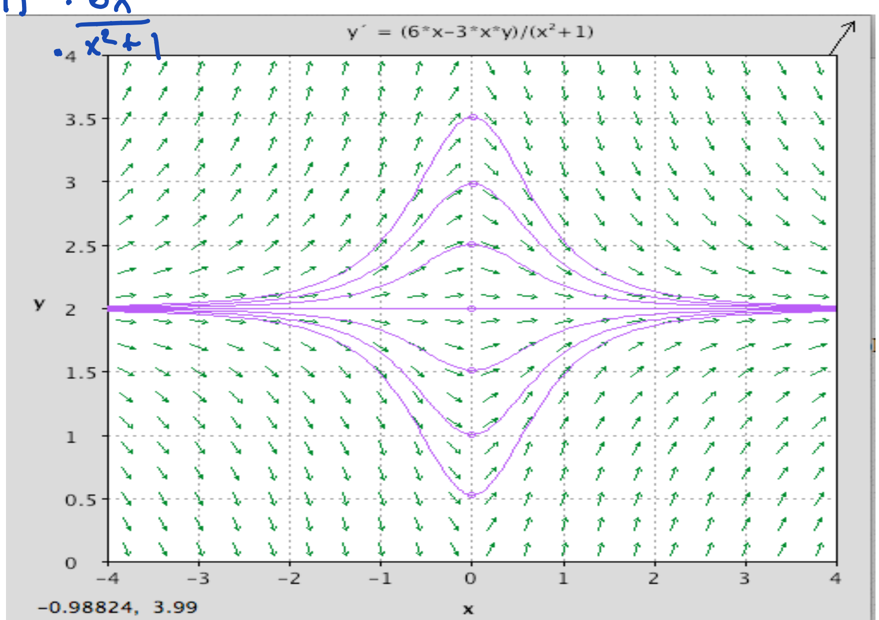
$(x^2+1)^{3/2}y = \int 6x(x^2+1)^{1/2}\,dx$

$(x^2+1)^{3/2}y = 2(x^2+1)^{3/2} + C$

$\frac{d}{dx}2(x^2+1)^{3/2} = \frac{3}{\cancel{2}}(x^2+1)^{1/2}\cdot\cancel{2}x$ $\quad -2$

⑤ $\div$ I.F.

$$\boxed{y = 2 + C\,(x^2+1)^{-3/2}}$$



$$\frac{dy}{dx} = \frac{3x}{x^2+1}(2-y)$$

$$* \quad \frac{dy}{y-2} = -\frac{3x}{x^2+1}\,dx \qquad \Big(y \neq 2 \to y(x) \equiv 2 \text{ is soln} \quad y'=0,\ RHS=0 \Big)$$

I'll fill in the rest!

filled in: integrate *

$$\int \frac{dy}{y-2} = \int \frac{-3x}{x^2+1}\,dx \longleftarrow \quad u = x^2+1, \quad du = 2x\,dx, \quad -\frac{3}{2}du = -3x\,dx$$

$$\ln|y-2| = -\frac{3}{2}\ln(x^2+1) + C \qquad \int \frac{-\frac{3}{2}du}{u} = -\frac{3}{2}\ln|u| = -\frac{3}{2}\ln(x^2+1)$$

$(u = y-2,\ du = dy)$

exponentiate:

$$|y-2| = e^{-\frac{3}{2}\ln(x^2+1)}\, e^{C_1}$$

$$y-2 = C(x^2+1)^{-3/2} \qquad C = \pm e^{C_1}$$

$$\boxed{y = 2 + C(x^2+1)^{-3/2}}$$

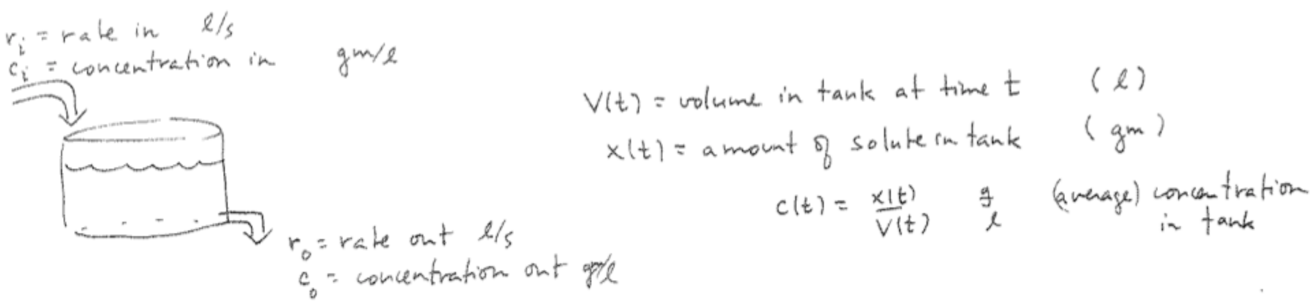
note $C=0$ yields the singular soln $y=2$

An extremely important class of modeling problems that lead to linear DE's involve input-output models. These have diverse applications ranging from bioengineering to environmental science. For example, The "tank" below could actually be a human body, a lake, or a pollution basin, in different applications.

For the present considerations, consider a tank holding liquid, with volume $V(t)$ (e.g units l). Liquid flows in at a rate r_i (e.g. units $\frac{l}{s}$), and with solute concentration c_i (e.g. units $\frac{gm}{l}$). Liquid flows out at a rate r_o , and with concentration c_0 . We are attempting to model the volume $V(t)$ of liquid and the amount of solute $x(t)$ (e.g. units gm) in the tank at time t , given $V(0) = V_0,\ x(0) = x_0$. We assume the solution in the tank is well-mixed, so that we can treat the concentration as uniform throughout the tank, i.e.

$$c_o = \frac{x(t)}{V(t)} \quad \frac{gm}{l} \ .$$

See the diagram below.



<u>Exercise 4:</u> Under these assumptions use your modeling ability and Calculus to derive the following differential equations for $V(t)$ and $x(t)$:
a) The DE for $V(t)$, which we can just integrate:

$$V'(t) = r_i - r_0$$

so $V(t) = V_0 + \int_0^t r_i(\tau) - r_0(\tau)\, d\tau$

b) The linear DE for $x(t)$.

$$x'(t) = r_i c_i - r_o c_o = r_i c_i - r_o \frac{x}{V}$$

$$x'(t) + \frac{r_o}{V} x(t) = r_i c_i$$

Often (but not always) the tank volume remains constant, i.e. $r_i = r_o$. If the incoming concentration c_i is also constant, then the IVP for solute amount is

$$x' + a\,x = b$$
$$x(0) = x_0$$

where a, b are constants. This differential equation is separable and linear, and it is recommended that you become good at solving it. Notice that it includes the exponential growth/decay and Newton's law of cooling DE's as special cases.