$= 2\int_0^a \frac{v_0}{v_s}\left(1-\frac{x^2}{a^2}\right)dx$ (integrand is even function)

$= \frac{2v_0}{v_s}\left[x - \frac{x^3}{3a^2}\right]_0^a = \frac{2v_0}{v_s}\left[\underbrace{a-\frac{a}{3}}_{2/3\,a}\right]$

$\boxed{= \frac{4v_0 a}{3v_s}}$ final answer

Math 2250-004
Wed Jan 11

- Quiz today at end of class, on section 1.1-1.2 material

- After finishing Tuesday's notes ~~if~~ is necessary, begin Section 1.3: slope fields and graphs of differential equation solutions: Consider the first order DE IVP for a function $y(x)$:

$$\boxed{y' = f(x, y) \ , y(x_0) = y_0 .}$$

for soln $y(x)$ its graph $y=y(x)$
$y'(x)$ is slope of graph, slope also given by $f(x,y)$

If $y(x)$ is a solution to this IVP and if we consider its graph $y = y(x)$, then the IC means the graph must pass through the point (x_0, y_0). The DE means that at every point (x, y) on the graph the slope of the graph must be $f(x, y)$. (So we often call $f(x, y)$ the "slope function" for the differential equation.) This gives a way of understanding the graph of the solution $y(x)$ even without ever actually finding a formula for $y(x)$! Consider a **slope field** near the point (x_0, y_0): at each nearby point (x, y), assign the slope given by $f(x, y)$. You can represent a slope field in a picture by using small line segments placed at representative points (x, y), with the line segments having slopes $f(x, y)$.

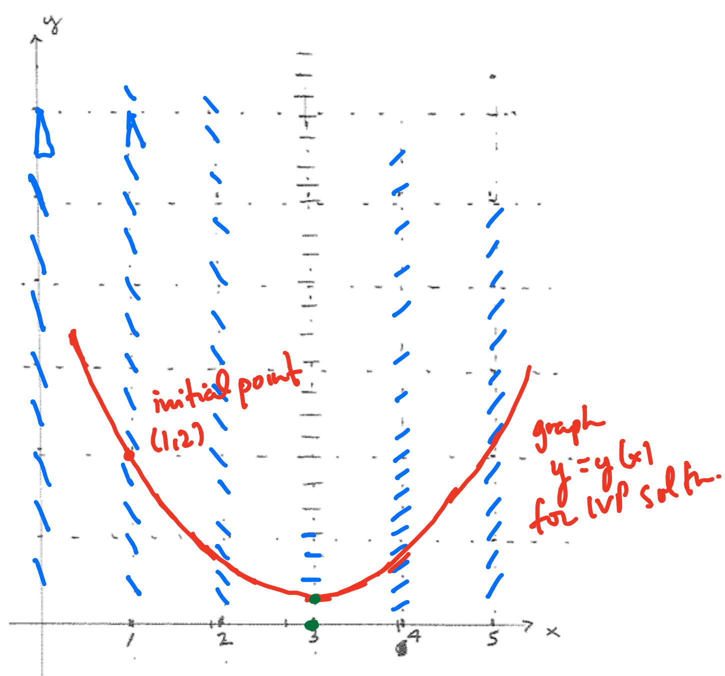
Exercise 1: Consider the differential equation $\frac{dy}{dx} = x - 3$, and then the IVP with $\boxed{y(1) = 2}$.
a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \le x \le 5, 0 \le y \le 6$. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.
b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.
c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).

a) $\frac{dy}{dx} = x-3$
sketch slope field.
§1.2 ex $f(x,y) = f(x)$
slope only depends on x

x	slope fcn $x-3$
$x=3$	0
4	1
2	-1
1	-2
0	-3
5	2
2.5	-.5

b) $\frac{dy}{dx} = x-3$
$y(1) = 2$

c) $y'(x) = x-3$
$y(x) = \int x-3\,dx$
$y = \frac{x^2}{2} - 3x + C$
$y(1) = 2$:
$2 = \frac{1}{2} - 3 + C$
$2 = -2\tfrac{1}{2} + C$
$\frac{9}{2} = 4\tfrac{1}{2} = C$
$y = \frac{x^2}{2} - 3x + \frac{9}{2}$
$= \frac{1}{2}(x^2 - 6x + 9) = \frac{1}{2}(x-3)^2$. Soln graph is $\underline{y = \frac{1}{2}(x-3)^2}$ vertex $(3,0)$



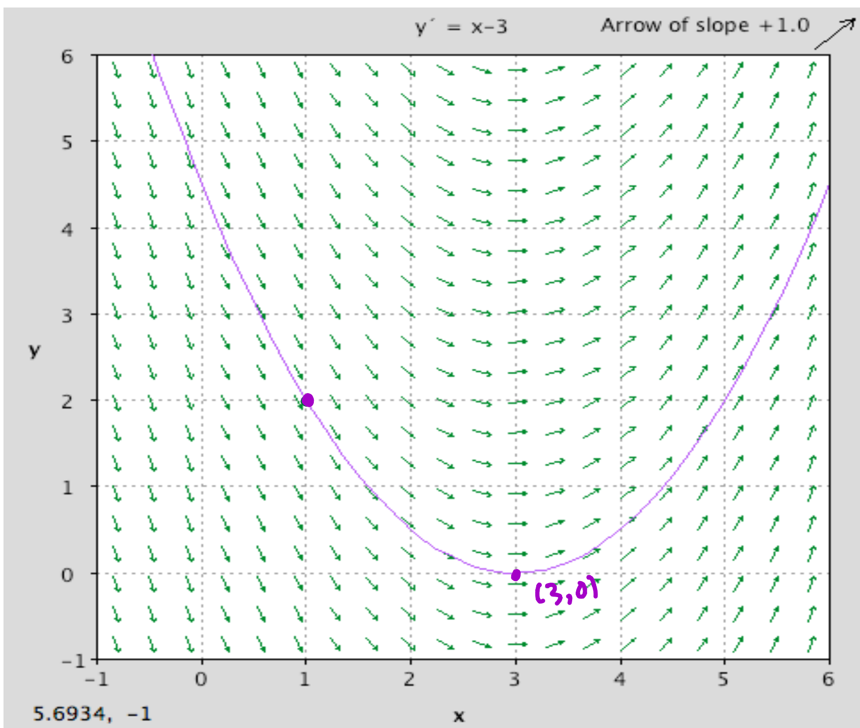
neg slopes, pos slopes

The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

$$\frac{\triangle y}{\triangle x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in x and y, starting from the initial point $\left(x_0, y_0\right)$. In this way one can approximate solution functions to IVPs, and their graphs. You can find an applet to do this by googling "dfield" (stands for "direction field", which is a synonym for slope field). Here's a picture like the one we sketched by hand on the previous page. The solution graph was approximated using numerical ideas as above, and this numerical technique works for much more complicated differential equations, e.g. when solutions exist but don't have closed form formulas. The program "dfield" was originally written for Matlab, and you can download a version to run inside that package. Or, you can download stand-alone java code.

google dfield
download java applet



Exercise 2: Consider the IVP

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

a) Check that $\boxed{y(x) = x + 1 + C\,e^{x}}$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

insert here.

$y(x) = x + 1 + Ce^x$

LHS $y' = 1 + Ce^x$

RHS $y - x = (\cancel{x} + 1 + Ce^x) - \cancel{x}$

LHS = RHS, so $y(x)$ is a soltn.

b) Solve the IVP by choosing appropriate C.

insert here

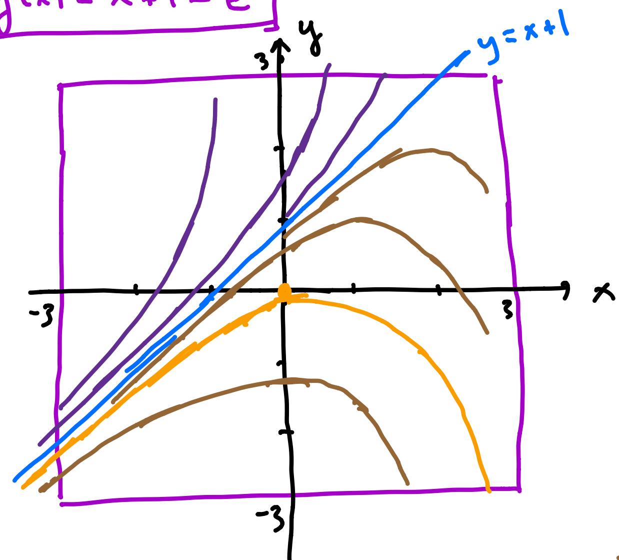
$y(0) = 0 + 1 + C \overset{\text{want}}{=} 0 \Rightarrow C = -1$

c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

d) Compare your work in (c) with the picture created by dfield on the next page.

b) soltn $\boxed{y(x) = x + 1 - e^x}$

c)

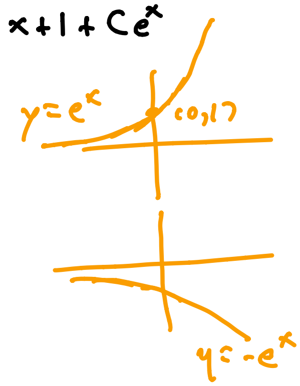


$y(x) = x + 1 + Ce^x$

$C < 0$

$C > 0$

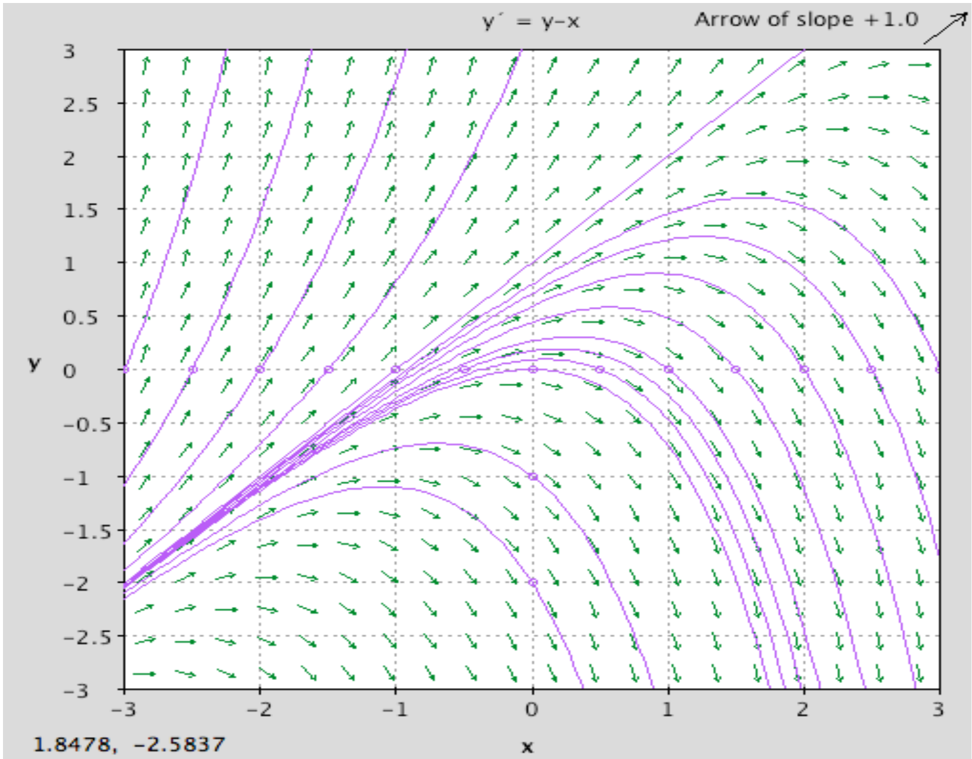
$y(x) = x + 1 + Ce^x$



$\lim_{x \to -\infty} e^x = 0$

$y' = 1 - e^x$

$y'' = -e^x$


y´ = y−x
Arrow of slope +1.0
y
x
3
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−3
−3
−2
−1
0
1
2
3
1.8478, −2.5837

- We'll use Wednesday's notes for a lot of the class
- I'll post a copy of the lab in our HW page
- our CANVAS page exists
- next week's notes should be posted by 3:00 today. Print these yourself for Tuesday class

Math 2250-004
Fri Jan 13

1.3-1.4: slope fields; existence and uniqueness for solutions to IVPs; examples we can check with separation of variables.

Exercise 1: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 .$$

not §1.2 problem! $y'(x) = f(x)$
$y(x) = \int f(x)dx$

a) Use separation of variables to find solutions to this DE...the "magic" algorithm that we talked about at the start of the week, but didn't explain the reasoning for. It is de-mystified on the next page of today's notes.
b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a)? (You can sketch by hand, I'll use "dfield" on my browser.)
c) Explain why each IVP has a solution, but this solution does not exist for all *x*.

You can download the java applet "dfield" from the URL
http://math.rice.edu/~dfield/dfpp.html
(You also have to download a toolkit, following the directions there.)

shorthand for
$y'(x) = 1 + y(x)^2$

BAD: $\frac{dy}{dx} = 1 + y^2$

$\int \frac{dy}{dx}\,dx = \int 1 + y^2\,dx$ $y(x)$

$y = x + ??$

don't know $y(x)$
don't know $y(x)^2$
so you can't compute
$\int y(x)^2 dx$

RIGHT WAY:

$\frac{1}{1+y^2}\frac{dy}{dx} = 1$

$\int \frac{dy}{1+y^2} = \int dx$!

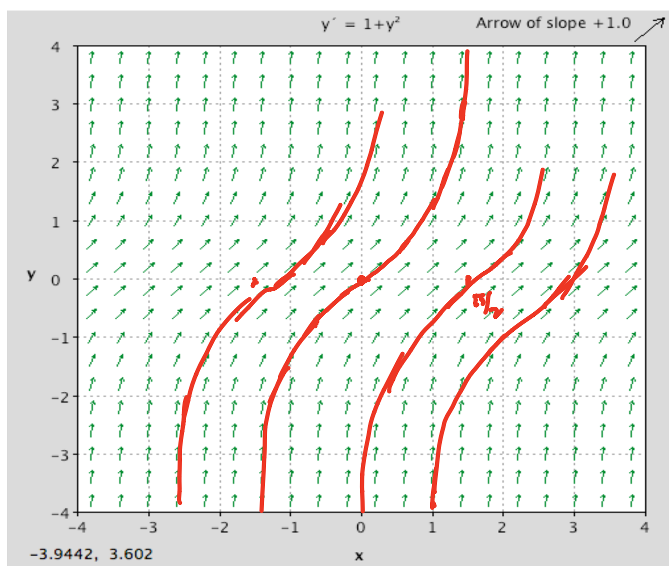
magic

$\arctan(y) = x + C$

$\tan(\arctan(y)) = \tan(x+C)$

$y = \tan(x+C)$

e.g. $C = 0$, $y = \tan x$ — vert. asym @ $x = \pm\frac{\pi}{2}$
solve IVP for $\begin{cases} y' = 1+y^2 \\ y(0) = 0 \end{cases}$



$y = \tan(x+C)$
related to
$y = \tan(x)$
shifted horizontally by $(-C)$

1.4 Separable DE's: Important applications, as well as a lot of the examples we study in slope field discussions of section 1.3 are separable DE's. So let's discuss precisely what they are, and why the separation of variables algorithm works.

Definition: A separable first order DE for a function $y = y(x)$ is one that can be written in the form:

$$\frac{dy}{dx} = f(x)\phi(y) .$$

It's more convenient to rewrite this DE as

$$\frac{1}{\phi(y)}\frac{dy}{dx} = f(x), \qquad (\text{as long as } \phi(y) \neq 0) .$$

Writing $g(y) = \dfrac{1}{\phi(y)}$ the differential equation reads

$$g(y)\frac{dy}{dx} = f(x) .$$

Solution (math justified): The left side of the modified differential equation is short for $g(y(x))\dfrac{dy}{dx}$. And if $G(y)$ is any antiderivative of $g(y)$, then we can rewrite this as

$$G'(y(x))y'(x)$$

which by the chain rule (read backwards) is nothing more than

$$\frac{d}{dx}G(y(x)).$$

And the solutions to

$$\frac{d}{dx}G(y(x)) = f(x)$$

are

$$G(y(x)) = \int f(x)\, dx = F(x) + C.$$

where $F(x)$ is any antiderivative of $f(x)$. Thus solutions $y(x)$ to the original differential equation satisify

$$G(y) = F(x) + C .$$

This expresses solutions $y(x)$ implicitly as functions of x. You may be able to use algebra to solve this equation explicitly for $y = y(x)$, and (working the computation backwards) $y(x)$ will be a solution to the DE. (Even if you can't algebraically solve for $y(x)$, this still yields implicitly defined solutions.)

Solution (differential magic): Treat $\dfrac{dy}{dx}$ as a quotient of differentials dy, dx, and multiply and divide the DE to "separate" the variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$g(y)dy = f(x)dx .$$

Antidifferentiate each side with respect to its variable (?!)

$$\int g(y)dy = \int f(x)dx, \text{ i.e.}$$

$$G(y) + C_1 = F(x) + C_2 \Rightarrow G(y) = F(x) + C . \quad \text{Agrees!}$$

This is the same differential magic that you used for the "method of substitution" in antidifferentiation, which was essentially the "chain rule in reverse" for integration techniques.

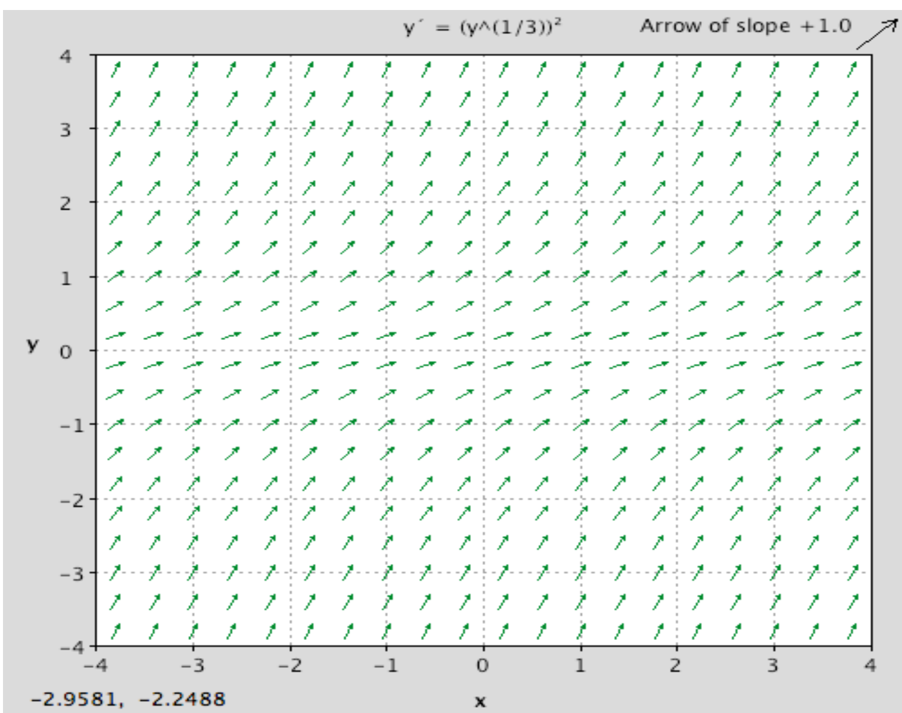
Exercise 2a) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$

$$y(0) = 0$$

2b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called singular solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

2c) Sketch some of these singular solutions onto the slope field below.



Here's what's going on (stated in 1.3 page 24 of text; partly proven in Appendix A.)
<u>Existence - uniqueness theorem for the initial value problem</u>
Consider the IVP

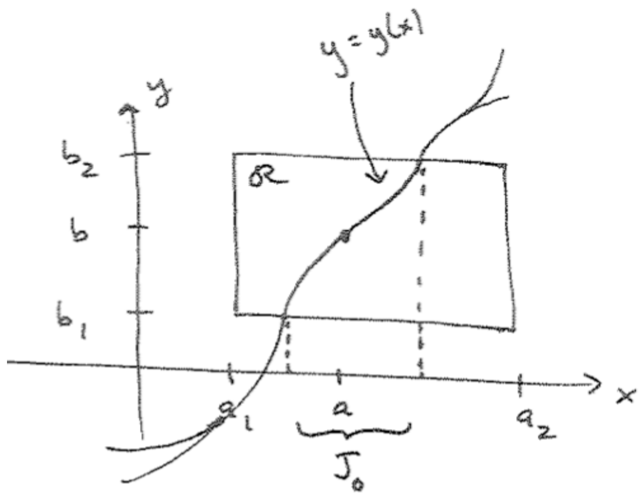
$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

• Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R}$: $a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the $x-y$ plane.

• <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.

• <u>Uniqueness:</u> If the partial derivative function $\frac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.



<u>Exercise 3</u>: Discuss how the existence-uniqueness theorem is consistent with our work in Wednesday's Exercises 1-2, and in today's Exercises 1-2 where we were able to find explicit solution formulas because the differential equations were actually separable.