Exercise 4) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate proportional to to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use this model to derive the differential equation

$$\frac{dT}{dt} = -k(T - A).$$

✓ ? $T'(t) = k_1(T-A)$

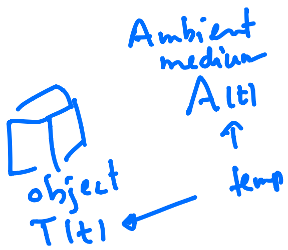
- if $T > A$, $T'(t) < 0$, $T-A > 0$ $\Rightarrow k_1 < 0$
- if $T < A$, $T'(t) > 0$, $T-A < 0$, $k_1 < 0$

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead?

Yes

c) Use this model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70 ° F. An hour later the body temperature has decreased to 60 °. It's been a winter inversion in SLC, with constant ambient temperature 30 °. Assuming the Newton's law model, estimate the time of death.

① model, DE
② DE, solve
③ apply.
c)



so $k_1 < 0$
so let's call it "$-k$" instead, $k > 0$
(we like positive constants)

solve $\frac{dT}{dt} = -k(T-A)$ const

$$\int \frac{dT}{T-A} = \int -k\, dt$$

$$\ln|T-A| = -kt + C$$

$$|T-A| = e^{-kt} e^{C}$$

$$T - A = Ce^{-kt}$$

$$\boxed{T(t) = A + Ce^{-kt}}$$

c): set $t=0$ to be 3:00 p.m.
use hour units for time

$T(0) = 70$
$T(1) = 60$
$A = 30$

$$T(t) = 30 + Ce^{-kt}$$

- find C: $T(0) = 70 = 30 + C \Rightarrow C = 40$
- find k: $T(1) = 60$ $60 = 30 + 40e^{-k}$
- set $T(t) = 98.6$ & solve for t.

$k = .2876$

$$T(t) = 30 + 40e^{-kt}$$

$$98.6 = 30 + 40e^{-kt}$$

solve for t: $t = -1.875$ hours.
($t=0$ was 3:00 p.m.)

$3 - 1.875 = 1.125$ o'clock

$\approx 1{:}07$

$.125 \times 60 = 7.50$

Math 2250-004
Tues Jan 10

Recall that course info - syllabus, class notes, homework, etc. is posted at our web page
http://www.math.utah.edu/~korevaar/2250spring17
There will also be course material posted on our CANVAS page.

• Recall from Monday that a 1^{st} order DE is an equation involving a function and its first derivative. We may chose to write the function and variable as $y = y(x)$. In this case the differential equation is an equation equivalent to one of the form

$$F(x, y, y') = 0 .$$

• We can often use algebra to solve for y', to get what we call the **standard form** for a first order DE:

$$y' = f(x, y) .$$

• If we want our solution function to a DE to also satisfy $y(x_0) = y_0$, and if our DE is written in standard form, then we say that we are solving an **initial value problem** (IVP):

$$\text{IVP} \quad \begin{cases} y' = f(x, y) & \leftarrow \text{DE} \\ y(x_0) = y_0 . & \leftarrow \text{IC} \end{cases}$$

With these ideas in mind, let's finish Monday's notes, including Exercises 3 and 4 (assuming we didn't finish them on Monday).

Tuesday notes on Wed.

Section 1.2: differential equations equivalent to ones of the form

$$y'(x) = f(x)$$

$\frac{dy}{dx} = f(x,y)$ in general

easiest case: what fcns $y(x)$ have derivatives $f(x)$? ans: antiderivatives!

which we solve by direct antidifferentiation

$$y(x) = \int f(x)\, dx = F(x) + C.$$

Exercise 1 Solve the initial value problem

$$\frac{dy}{dx} = x\sqrt{x^2 + 4}$$
$$y(0) = 0$$

(a) solve the DE

$$y = \int x\sqrt{x^2+4}\, dx$$

$u = x^2+4$
$du = 2x\, dx$
$\frac{1}{2}du = x\, dx$

$$= \int u^{1/2}\, \tfrac{1}{2}du$$
$$= \frac{1}{2}\int u^{1/2}\, du$$
$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$
$$= \frac{1}{3}u^{3/2} + C$$
$$\boxed{y = \frac{1}{3}(x^2+4)^{3/2} + C}$$

(b) IVP:

$$y(0) = 0 = \frac{1}{3}\cdot 4^{3/2} + C$$
$$0 = \frac{1}{3}8 + C$$
$$-8/3 = C$$

IVP $\boxed{y = \frac{1}{3}(x^2+4)^{3/2} - 8/3}$

An important class of such problems arises in physics usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) \, .$$

$m\,x''(t) = $ net forces
if
$= f(t)$ then §1.2

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

Exercise 2:

a) If the units for position are meters m and the units for time are seconds s, what are the units for velocity and acceleration? (These are *mks* units.) vel. units m/s ; accel m/s^2

b) Same question, if we use the English system in which length is measure in feet and time in seconds. Could you convert between *mks* units and English units?

velocity units ft/sec
accel. unit ft/sec^2

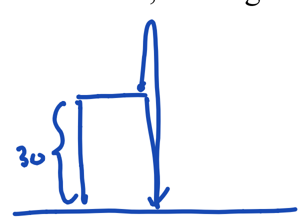
units $x(t)$ length

$v(t) = x'(t) = \lim_{\Delta t \to 0} \dfrac{x(t+\Delta t) - x(t)}{\Delta t} \leftarrow \dfrac{\text{length}}{\text{time}}$

$v'(t) = x''(t) = \lim_{\Delta t \to 0} \dfrac{v(t+\Delta t) - v(t)}{\Delta t} \quad \dfrac{\text{length/time}}{\text{time}} = \dfrac{\text{length}}{\text{time}^2}$

Exercise 3: A projectile with very low air resistance is fired almost straight up from the roof of a building 30 meters high, with initial velocity 50 m/s. Its initial horizontal velocity is near zero, but large enough so that the object lands on the ground rather than the roof.

a) Neglecting friction, how high will the object get above ground?
b) When does the object land ?



(find formulas for ht & velocity).

Let $y(t)$ be height @ time t (m)
$y(0) = 30$, (choice to set ground level as $y=0$)
$v(0) = y'(0) = 50$ m/s

$m\,y''(t) = -mg$ $\quad g \approx 9.8$ m/s^2

$\boxed{y''(t) = -g}$ Let's Ex 4 at this point

transpose Ex 4:

$\boxed{y(t) = -\tfrac{1}{2}g t^2 + v_0 t + y_0}$

$\boxed{v(t) = -gt + v_0}$

$\boxed{\begin{array}{l} y(t) = -4.9t^2 + 50t + 30 \\ v(t) = -9.8t + 50 \end{array}}$

a) how high?
set $y'(t) = 0$
$v(t) = 0$
solve for t (when)
$-9.8t + 50 = 0$
$\boxed{t = 5.1}$ sec
height $\boxed{y(5.1) = 157.55 \text{ m}}$

b) When does it land.
Solve $y(t) = 0$ for t
$\boxed{t = 10.76 \text{ sec}}$

Exercise 4:

Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a .$$

(This could happen for vertical motion, e.g. near the earth's surface with $a = g \approx 9.8\ \frac{m}{s^2} \approx 32\ \frac{ft}{s^2}$, as well as in other situations.)

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{max} = \frac{1}{2}\frac{v_0^2}{a} .$$

(This formula may help with some homework or lab problems, besides being interesting.)

a)

$$x''(t) = -a$$

$$x'(t) = \int -a\,dt = -at + C$$

$$v(t) = -at + C$$

$$@\ t=0:\ v(0) = 0 + C \Rightarrow C = v_0$$

$$\int x'(t) = \boxed{v(t) = -at + v_0}$$

$$\Rightarrow x(t) = \int -at + v_0\,dt$$

$$x(t) = -\tfrac{1}{2}at^2 + v_0 t + C$$

$$@\ t=0:\ x_0 = 0 + 0 + C. \Rightarrow x_0 = C$$

$$\boxed{x(t) = -\tfrac{1}{2}at^2 + v_0 t + x_0}$$

Here's another fun example from section 1.2, which also reviews important ideas from Calculus - in particular we will see how the fact that the slope of a graph $y = g(x)$ is the derivative $\frac{dy}{dx}$ can lead to first order differential equations.

Exercise 5: (See text, page 16). A swimmer wishes to cross a river of width $w = 2\,a$, by swimming directly towards the opposite side, with constant transverse velocity v_S. The river velocity is fastest in the middle and is given by an even function of x, for $-a \le x \le a$. The velocity equal to zero at the river banks. For example, it could be that

$$\boxed{v_R(x) = v_0\left(1 - \frac{x^2}{a^2}\right)}.$$

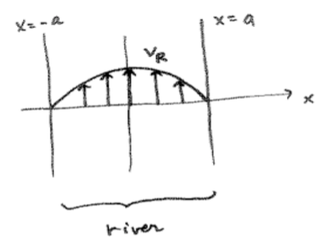
See the configuration sketches below.
a) Writing the swimmer location at time t as $(x(t), y(t))$, translate the information above into expressions for $x'(t)$ and $y'(t)$.
b) The parametric curve describing the swimmer's location can also be expressed as the graph of a function $y = y(x)$. Show that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right).$$

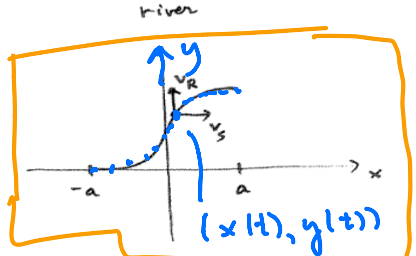
Filled in after class on Wed

c) Compute an integral or solve a DE, to figure out how far downstream the swimmer will be when she reaches the far side of the river.

set up: a) $x'(t) = v_s$ const
$y'(t) = v_R(x(t)) = v_0\left(1 - \frac{x^2}{a^2}\right)$



b) the parametric curve of locations at time t, i.e. $t \to (x(t), y(t))$ can also be thought of as a graph $y = y(x)$ (where $x = x(t)$) so by Calc 1,



from (a)

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt} \Rightarrow \underbrace{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_0(1 - x^2/a^2)}{v_s} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)}_{\frac{dy}{dx} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)\ \checkmark}$$

c) from orange configuration we want $y(a) - y(-a)$ where $y(x)$ solves IVP $\begin{cases} \frac{dy}{dx} = \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right) \\ y(-a) = 0 \leftarrow \text{our choice of origin on y-axis} \end{cases}$

Could solve IVP at right, or just use Fund. Theorem of Calc:

$$y(a) - y(-a) = \int_{-a}^{a} y'(x)\,dx = \int_{-a}^{a} \frac{v_0}{v_s}\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= 2\int_0^a \frac{v_0}{v_s}\left(1-\frac{x^2}{a^2}\right)dx$$ (integrand is even function)

$$= \frac{2v_0}{v_s}\left[x-\frac{x^3}{3a^2}\right]_0^a = \frac{2v_0}{v_s}\left[\underbrace{a-\frac{a}{3}}_{2/3\,a}\right]$$

$$\boxed{=\frac{4v_0a}{3v_s}}$$ final answer

Math 2250-004
Wed Jan 11

- Quiz today at end of class, on section 1.1-1.2 material

- After finishing Tuesday's notes ~~if~~ is necessary, begin Section 1.3: slope fields and graphs of differential equation solutions: Consider the first order DE IVP for a function $y(x)$:

$$y' = f(x, y) \quad , y(x_0) = y_0 .$$

If $y(x)$ is a solution to this IVP and if we consider its graph $y = y(x)$, then the IC means the graph must pass through the point (x_0, y_0). The DE means that at every point (x, y) on the graph the slope of the graph must be $f(x, y)$. (So we often call $f(x, y)$ the "slope function" for the differential equation.) This gives a way of understanding the graph of the solution $y(x)$ even without ever actually finding a formula for $y(x)$! Consider a **slope field** near the point (x_0, y_0): at each nearby point (x, y), assign the slope given by $f(x, y)$. You can represent a slope field in a picture by using small line segments placed at representative points (x, y), with the line segments having slopes $f(x, y)$.

Exercise 1: Consider the differential equation $\frac{dy}{dx} = x - 3$, and then the IVP with $y(1) = 2$.

a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \le x \le 5, 0 \le y \le 6$. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started, see below.
b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without resorting to a formula for the solution function $y(x)$.
c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).