Exercise 2 Coefficient matrix taken from problem #19, section 3.3, page 174.

```
> with(LinearAlgebra) :
> A := Matrix(3, 5, [2, 7, -10, -19, 13,
                     1, 3, -4, -8, 6,
                     1, 0, 2, 1, 3]);
```

$$A := \begin{bmatrix} 2 & 7 & -10 & -19 & 13 \\ 1 & 3 & -4 & -8 & 6 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \tag{3}$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \tag{4}$$

Let's consider three different linear systems for which A is the coefficient matrix. In the first one, the right hand sides are all zero (what we call the "homogeneous" problem), and I have carefully picked the other two right hand sides. The three right hand sides are separated by the dividing line below:

$$\left[\begin{array}{ccccc|c|c|c} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right].$$

$\uparrow b1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\uparrow b2 = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ $\uparrow b3 = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$

e.g. 3rd system

$2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 = 7$

$x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 = 3$

$x_1 + 2x_3 + x_4 + 3x_5 = 0$

$A\vec{x} = \vec{b3}$

We'll try solving three linear systems at once!

```
> b1 := Vector([0, 0, 0]) :
  b2 := Vector([7, 0, 0]) :
  b3 := Vector([7, 3, 0]) :
  C := ⟨A|b1|b2|b3⟩; # very augmented matrix
  ReducedRowEchelonForm(C);
```

$$C := \left[\begin{array}{ccccc|c|c|c} 2 & 7 & -10 & -19 & 13 & 0 & 7 & 7 \\ 1 & 3 & -4 & -8 & 6 & 0 & 0 & 3 \\ 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \end{array}\right]$$

$x_2 - 2x_3 - 3x_4 - x_5 = 0 \longrightarrow$

$$\left[\begin{array}{ccccc|c|c|c} 1 & 0 & 2 & 1 & 3 & 0 & 0 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{5}$$

$x_2 - 2x_3 - 3x_4 + x_5 = 1$

2a) Find the solution sets for each of the three systems, using the reduced row echelon form of C.

$A\vec{x} = \vec{b1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 2t_3 + 3t_4 - t_5$

$x_3 = t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$

$A\vec{x} = \vec{b2}$

$\Downarrow$

no solutions:

last row "says"

$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 1$

$0 = 1$

no $\vec{x}$ can make this true.

$A\vec{x} = \vec{b3} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$

$x_1 = -2t_3 - t_4 - 3t_5$

$x_2 = 1 + 2t_3 + 3t_4 - t_5$

$x_3 = t_3 \in \mathbb{R}$

$x_4 = t_4 \in \mathbb{R}$

$x_5 = t_5 \in \mathbb{R}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t_3 \\ 2t_3 \\ t_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t_4 \\ 3t_4 \\ 0 \\ t_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3t_5 \\ -t_5 \\ 0 \\ 0 \\ t_5 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{t_3 \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{\text{same}}} + t_5 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad *$$

$$\vec{x} = \vec{\text{same}} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad *$$

Important conceptual questions:

2b) Which of these three solutions could you have written down just from the reduced row echelon form of A, i.e. without using the augmented matrix and the reduced row echelon form of the augmented matrix? Why?

$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$: for $A \left| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$ $\to$ $\text{rref}(A) \left| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\}$

augmented matrix reduces to a matrix whose last column is also all zeroes

2c) Linear systems in which right hand side vectors equal zero are called <u>homogeneous</u> linear systems. Otherwise they are called <u>inhomogeneous</u> or <u>nonhomogeneous.</u> Notice that the general solution to the consistent inhomogeneous system is the sum of a particular solution to it, together with the general solution to the homogeneous system!!! Was this an accident? It's related to an important general concept which will keep coming up in the rest of the course. NO ?

2d) Can you tell how many free parameters the solutions to a matrix system $A\,\boldsymbol{x} = \boldsymbol{b}$ will have, based on the reduced row echelon form of A alone (assuming the system is consistent, i.e. has at least one solution)?

yes: # of free params = # of columns without Leading 1's.

Exercise 3) The reduced row echelon form of a (non-augmented) matrix A can tell us a lot about the possible solution sets to linear systems with augmented matrices $\langle A|\underline{\boldsymbol{b}}\rangle$.

First (notation), recall that linear systems

$$\begin{array}{ccccccccc} a_{11}x_1 &+& a_{12}x_2 &+\ldots+& a_{1n}x_n &=& b_1 \\ a_{21}x_1 &+& a_{22}x_2 &+\ldots+& a_{2n}x_n &=& b_2 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 &+& a_{m2}x_2 &+\ldots+& a_{mn}x_n &=& b_m \end{array}$$

can be written more efficiently using the rule we use to multiply a matrix times a vector,

$$\boldsymbol{A}\,\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

so that they system above can be abbreviated by $\boxed{A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}}$.

$$\begin{bmatrix} 1 & -1 & 2 \\ 4 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 - 1\cdot 5 + 2\cdot 7 \\ 4\cdot 0 + 0\cdot 5 + 3\cdot 7 \\ 2\cdot 0 - 1\cdot 5 + 1\cdot 7 \end{bmatrix} = \begin{bmatrix} 9 \\ 21 \\ 2 \end{bmatrix}$$

Then consider the matrix A below, and answer all questions:

```
> A := Matrix(2, 5, [2, 7,-10,-19, 13, 1, 3,-4,-8, 6]);
  ReducedRowEchelonForm(A);
```

2 eqtns in 5 unknowns.

$$A := \left[\begin{array}{ccccc|c} 2 & 7 & -10 & -19 & 13 & 0 \\ 1 & 3 & -4 & -8 & 6 & 0 \end{array}\right] \begin{array}{c} b_1 \\ b_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 3 & 0 \\ 0 & 1 & -2 & -3 & 1 & 0 \end{array}\right] \begin{array}{c} c_1 \\ c_2 \end{array} \qquad (6)$$

$x_2 - 2x_3 - 3x_4 + x_5 = 0$

$x_1 = -2t_3 - t_4 - 3t_5$
$x_2 = 2t_3 + 3t_4 - t_5$
$x_3 = t_3 \in \mathbb{R}$
$x_4 = t_4 \in \mathbb{R}$
$x_5 = t_5 \in \mathbb{R}$

3a) Is the homogeneous problem $A\underline{\boldsymbol{x}}=\underline{\boldsymbol{0}}$ ~~always~~ solvable? YES
"does it have solutions"

3b) Is the inhomogeneous problem $A\underline{\boldsymbol{x}}=\underline{\boldsymbol{b}}$ solvable no matter the choice of $\underline{\boldsymbol{b}}$?

3c) How many solutions are there? How many free parameters are there in the solution? How does this number relate to the reduced row echelon form of A?

3a) actually for any matrix A, $A\vec{x}=\vec{0}$ always has $\vec{x}=\vec{0}$ as a sol'n.

3b) Yes, no problem backsolving here.

3c) ∞'ly many sol'ns, 3 free parameters (3 columns w/o leading 1's in the reduced r.e.f. of A)

Exercise 4) Now consider the matrix B and similar questions:

> B := Matrix(3, 2, [1, 2, -1, 3, 4, 2]);
> ReducedRowEchelonForm(B);

$$B := \left[\begin{array}{cc|c:c} 1 & 2 & 0 & b_1 \\ -1 & 3 & 0 & b_2 \\ 4 & 2 & 0 & b_3 \end{array}\right]$$

coeff matrix for system of 3 equations in 2 unknowns, x_1, x_2

$$\left[\begin{array}{cc|c:c} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 0 & c_3 \end{array}\right] \quad (7)$$

each c_i is some sum of multiples of b_1, b_2, b_3

4a) How many solutions to the homogeneous problem $B\underline{\mathbf{x}} = \underline{\mathbf{0}}$?
4b) Is the inhomogeneous problem $B\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ solvable for every right side vector $\underline{\boldsymbol{b}}$?
4c) When the inhomogeneous problem is solvable, how many solutions does it have?

4a) backsolve the reduced augmented matrix

$x_1 = 0$
$x_2 = 0$
$0 = 0$

4b) if $c_3 \neq 0$, 3rd eqtn says $0x_1 + 0x_2 = c_3$ NO SOLUTIONS

(why aren't the c_3's always zero, no matter what $\underline{b}$ was?
Ans: pick $c_3 \neq 0$, reverse the elem. row ops, on augmented matrix. That'll produce a $\underline{b}$ that didn't have a sol'n.)

4c) if $c_3 = 0$, backsolve:

$x_1 = c_1$
$x_2 = c_2$
$0 = 0$

one sol'n

Exercise 5) Square matrices (i.e number of rows equals number of columns) with 1's down the diagonal which runs from the upper left to lower right corner are special. They are called identity matrices, I (because $I\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}$ is always true (as long as the vector $\underline{\boldsymbol{x}}$ is the right size)).

```
> C := Matrix(4, 4, [1, 0, -1, 1, 22, -1, 3, 5, 7, 4, 6, 2, 3, 5, 7, 13]);
  ReducedRowEchelonForm(C);
```

$$C := \left[\begin{array}{cccc|c|c} 1 & 0 & -1 & 1 & 0 & b_1 \\ 22 & -1 & 3 & 5 & 0 & b_2 \\ 7 & 4 & 6 & 2 & 0 & b_3 \\ 3 & 5 & 7 & 13 & 0 & b_4 \end{array}\right] \quad \text{reduce.}$$

$$\to \left[\begin{array}{cccc|c|c} 1 & 0 & 0 & 0 & 0 & c_1 \\ 0 & 1 & 0 & 0 & 0 & c_2 \\ 0 & 0 & 1 & 0 & 0 & c_3 \\ 0 & 0 & 0 & 1 & 0 & c_4 \end{array}\right] \qquad (8)$$

`>`

5a) How many solutions to the homogeneous problem $C\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$?
5b) Is the inhomogeneous problem $C\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ solvable for every choice of $\underline{\boldsymbol{b}}$?
5c) How many solutions?

5a) 1 soltn. $\vec{x} = \vec{0}$
5b) 1 soltn.

$x_1 = c_1$
$x_2 = c_2$
$x_3 = c_3$
$x_4 = c_4$

Exercise 6: What are your general conclusions?

6a) What conditions on the reduced row echelon form of the matrix A guarantee that the homogeneous equation $A\underline{x} = \underline{0}$ has infinitely many solutions? ($\vec{x} = \vec{0}$ is always one sol'n to $A\vec{x} = \vec{0}$)

6b) What conditions on the dimensions of A (i.e. number of rows and number of columns) always force infinitely many solutions to the homogeneous problem?

6c) What conditions on the reduced row echelon form of A guarantee that solutions $\underline{x}$ to $A\underline{x} = \underline{b}$ are always unique (if they exist)?

6d) If A is a square matrix ($m=n$), what can you say about the solution set to $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ when

- The reduced row echelon form of A is the identity matrix?
- The reduced row echelon form of A is not the identity matrix?

Solve $A\vec{x} = \vec{b}$

$$m\left\{\underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{n}\right._{m\times n}, \quad \vec{x}\in\mathbb{R}^n, \ \vec{b}\in\mathbb{R}^m$$

$A_{m\times n}$ means m rows, n columns

i.e. syst. of m equations in n unknowns.

6a) ∞'ly many sol'ns

same as free parameters, i.e. at least 1 column in rref(A) without a (row) leading 1

6b) more columns than rows in $A_{m\times n}$

$n > m$

6c) no free parameters

i.e. every column of r.r.e.f. of A has a leading 1

6d) $\text{rref}(A) = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \Rightarrow$ 1 solution.

$\text{rref}(A) \neq I \Longrightarrow$ at least one row of zeroes in rref(A) & at least one free param (when sol'ns exist)

(so either no solutions or infinitely many)

• Finish Tuesday's notes about what rref (A) implies about solutions $\vec{x}$ to $A\vec{x} = \vec{b}$
• then part way through today's notes
• quiz.

Wed Feb 8
3.4 Matrix algebra

• Our first exam is next Friday February 17. Your homework assignment due next Wednesday Feb 15 covers 3.4-3.6. The exam will cover through 3.6.

• Vector and matrix algebra, section 3.4:

Matrix vector algebra that we've already touched on, but that we want to record carefully:

Vector addition and scalar multiplication:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ : \\ x_n + y_n \end{bmatrix} ; \qquad c \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} c\,x_1 \\ c\,x_2 \\ c\,x_3 \\ : \\ c\,x_n \end{bmatrix}$$

Vector dot product , which yields a scalar (i.e. number) output (regardless of whether vectors are column vectors or row vectors):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ : \\ y_n \end{bmatrix} := x_1\,y_1 + x_2\,y_2 + ... + x_n\,y_n .$$

Matrix times vector: If A is an $m \times n$ matrix and $\underline{\boldsymbol{x}}$ is an n column vector, then

$$A\underline{\boldsymbol{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & ... & a_{1\,n} \\ a_{21} & a_{22} & a_{23} & ... & a_{2\,n} \\ : & & & & : \\ a_{m1} & a_{m2} & a_{m3} & ... & a_{m\,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ : \\ x_n \end{bmatrix} := \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1\,n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2\,n}x_n \\ : \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{m\,n}x_n \end{bmatrix} = \begin{bmatrix} Row_1(A) \cdot \underline{\boldsymbol{x}} \\ Row_2(A) \cdot \underline{\boldsymbol{x}} \\ : \\ Row_m(A) \cdot \underline{\boldsymbol{x}} \end{bmatrix} = \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Compact way to write our usual linear system:

$$A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}} .$$