Solutions to linear equations in 3 unknowns:

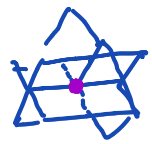
What is the geometric question you're answering?] common (intersection) points of a collection of planes

Exercise 4) Consider the system

$$x + 2y + z = 4$$
$$3x + 8y + 7z = 20$$
$$2x + 7y + 9z = 23\,.$$

Use elementary equation operations (or if you prefer, elementary row operations in the synthetic version) to find the solution set to this system. There's a systematic way to do this, which we'll talk about. It's called Gaussian elimination.

Hint: The solution set is a single point, $\boxed{[x, y, z] = [5, -2, 3]}$.



$$\begin{array}{ccc|c} -3\cdot 1 & -1\cdot 2 & -3\cdot 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}$$

$$\begin{array}{r|ccc|c} & 1 & 2 & 1 & 4 \\ -3R_1+R_2 & 0 & 2 & 4 & 8 \\ -2R_1+R_3 & 0 & 3 & 7 & 15 \end{array}$$

$$\begin{array}{r|ccc|c} & 1 & 2 & 1 & 4 \\ R_2/2 & 0 & 1 & 2 & 4 \\ & 0 & 3 & 7 & 15 \end{array}$$

$$\begin{array}{r|ccc|c} & 1 & 2 & 1 & 4 \\ & 0 & 1 & 2 & 4 \\ -3R_2+R_3 & 0 & 0 & 1 & 3 \end{array} \longrightarrow$$

backsolve

$z = 3$

$\Rightarrow y + 2z = 4 \Rightarrow y = 4 - 2z = -2$

$\Rightarrow x + 2(-2) + 1\cdot 3 = 4 \Rightarrow x = 5$

$$\begin{array}{r|ccc|c} -R_3+R_1 & 1 & 2 & 0 & 1 \\ -2R_3+R_2 & 0 & 1 & 0 & -2 \\ & 0 & 0 & 1 & 3 \end{array}$$

$$\begin{array}{r|ccc|c} -2R_2+R_1 & 1 & 0 & 0 & 5 \\ & 0 & 1 & 0 & -2 \\ & 0 & 0 & 1 & 3 \end{array} \xrightarrow{\text{"uncloak"}}$$

$x = 5$
$y = -2$
$z = 3$

Exercise 5 There are other possibilities. In the two systems below we kept all of the coeffients the same as in Exercise 4, except for a_{33} , and we changed the right side in the third equation, for 4a. Work out what happens in each case.

5a)

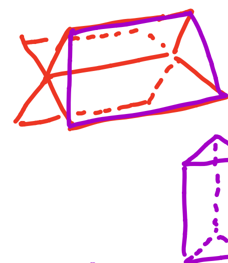
$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 20\,. \end{aligned}$$

5b)

$$\begin{aligned} x + 2y + z &= 4 \\ 3x + 8y + 7z &= 20 \\ 2x + 7y + 8z &= 23\,. \end{aligned}$$

5c) What are the possible solution sets (and geometric configurations) for 1, 2, 3, 4,... equations in 3 unknowns?

& geometric.



$$\left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 3 & 8 & 7 & 20 & 20 \\ 2 & 7 & 8 & 20 & 23 \end{array}\right]$$

$$\begin{array}{r} \\ -3R_1+R_2 \\ -2R_1+R_3 \end{array} \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 2 & 4 & 8 & 8 \\ 0 & 3 & 6 & 12 & 15 \end{array}\right]$$

$$\begin{array}{r} \\ R_2/2 \\ R_3/3 \end{array} \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 1 & 2 & 4 & 5 \end{array}\right]$$

$$\begin{array}{r} \\ \\ -R_2+R_3 \end{array} \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 4 & 4 \\ 0 & 1 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} -2R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccc|cc} 1 & 0 & -3 & -4 & * \\ 0 & 1 & 2 & 4 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

5a

$$\left.\begin{aligned} x + 2y + z &= 4 \\ y + 2z &= 4 \\ 0 &= 0 \end{aligned}\right\}$$

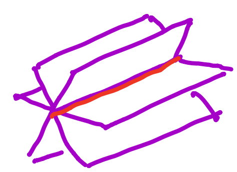
5b

$$\begin{aligned} x + 2y + z &= 4 \\ y + 2z &= 4 \\ 0 &= 1 \end{aligned}$$

no solutions!
no choice of x, y, z
will ever make 0 = 1.
so orig. sys had no solns!

5a

$$\begin{aligned} x \quad - 3z &= -4 \\ y + 2z &= 4 \end{aligned}$$



$$\begin{aligned} x &= -4 + 3t \\ y &= 4 - 2t \\ z &= t \in \mathbb{R}^3 \end{aligned}$$

in 2210
(1320?).

$$\vec{r}(t) = (-4+3t)\,\hat{i} + (4-2t)\,\hat{j} + t\,\hat{k}\,.$$

$$= (-4+3t)\begin{bmatrix}1\\0\\0\end{bmatrix} + (4-2t)\begin{bmatrix}0\\1\\0\end{bmatrix} + t\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}-4+3t\\4-2t\\t\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-4+3t\\4-2t\\t\end{bmatrix} = \begin{bmatrix}-4\\4\\0\end{bmatrix} + \begin{bmatrix}3t\\-2t\\t\end{bmatrix}$$

$$= \begin{bmatrix}-4\\4\\0\end{bmatrix} + t\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

soln set is a line! at $t=0$, $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-4\\4\\0\end{bmatrix}$, "velocity" vector is $\begin{bmatrix}3\\-2\\1\end{bmatrix}$

• Gaussian elimination
• "reduced row echelon form"
• Start with Friday's notes

Math 2250-004
Week 5 notes, February 6-10; Sections 3.2-3.5
3.2 Gaussian elimination for solving systems of linear algebraic equations
3.3 Structure of solutions sets for systems of linear equations
3.4 Matrix algebra
3.5 Matrix inverses

Mon Feb 6
<u>3.1-3.3</u> Linear systems of (algebraic) equations and how to solve them via Gaussian elimination and the reduced row echelon form of augmented matrices.

• Discuss any remaining parts of last Friday's notes. As we solve those systems of linear algebraic equations we begin to see how to systematically approach the problem of finding the explicit solution set. The precise details are below, and they should make good sense after the Friday examples. There is also a larger example in today's notes.

....
Summary of the systematic method known as Gaussian elimination:

We write the linear system (LS) of m equations for the vector $\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_n]^T$ of the n unknowns as

$$\begin{array}{rcl} a_{11}\,x_1 + a_{12}\,x_2 + \ldots + a_{1\,n}\,x_n &=& b_1 \\ a_{21}\,x_1 + a_{22}\,x_2 + \ldots + a_{2\,n}\,x_n &=& b_2 \\ \vdots \qquad\qquad\qquad\quad \vdots && \vdots \\ a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots + a_{m\,n}\,x_n &=& b_m \end{array}$$

m linear eqtns
in
n unknowns

The matrix that we get by adjoining (augmenting) the right-side $\underline{\boldsymbol{b}}$-vector to the coefficient matrix $A = [a_{i\,j}]$ is called the <u>*augmented matrix*</u> $\langle A|\underline{\boldsymbol{b}}\rangle$:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \ldots & a_{1\,n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2\,n} & b_2 \\ \vdots & & & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{m\,n} & b_m \end{array}\right]$$

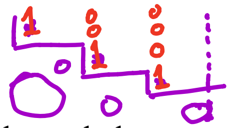
you might have more than one right-side vector $\vec{b}$ at once.

Our goal is to find all the solution vectors $\underline{\boldsymbol{x}}$ to the system - i.e. the <u>*solution set*</u> .

There are three types of <u>*elementary equation operations*</u> that don't change the solution set to the linear system. They are
- interchange two of equations
- multiply one of the equations by a non-zero constant
- replace an equation with its sum with a multiple of a different equation.

And that when working with the augmented matrix $\langle A|\underline{\boldsymbol{b}}\rangle$ these correspond to the three types of <u>*elementary row operations:*</u>
- interchange ("swap") two rows
- multiply one of the rows by a non-zero constant
- replace a row by its sum with a multiple of a different row.

Gaussian elimination: Use elementary row operations and work column by column (from left to right) and row by row (from top to bottom) to first get the augmented matrix for an equivalent system of equations which is in



row-echelon form:

(1) All "zero" rows (having all entries $= 0$) lie beneath the non-zero rows.

(2) The leading (first) non-zero entry in each non-zero row lies strictly to the right of the one above it.

(At this stage you could "backsolve" to find all solutions.)

Next, continue but by working from bottom to top and from right to left instead, so that you end with an augmented matrix that is in

reduced row echelon form: (1),(2), together with

(3) Each leading non-zero row entry has value 1 . Such entries are called "leading $1's$ "

(4) Each column that has (a row's) leading 1 has $0's$ in all the other entries.

Finally, read off how to explicitly specify the solution set, by "backsolving" from the reduced row echelon form.

Note: There are lots of row-echelon forms for a matrix, but only one reduced row-echelon form. All mathematical software will have a command to find the reduced row echelon form of a matrix.

Exercise 1 Find all solutions to the system of 3 linear equations in 5 unknowns

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 &= 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 &= 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 &= 27 . \end{aligned}$$

Here's the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right]$$

Find the reduced row echelon form of this augmented matrix and then backsolve to explicitly parameterize the solution set. (Hint: it's a two-dimensional plane in $\mathbb{R}^5$, if that helps. :-))

$$\left.\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{array}\right.$$

$$\begin{array}{r} \\ -2R_1+R_2 \\ -3R_1+R_3 \end{array} \left.\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{array}\right.$$

$$\begin{array}{r} \\ R_3 \\ R_2 \end{array} \left.\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & -1 & 8 & -13 \end{array}\right.$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{array}\right] \begin{array}{l} \\ \\ -2R_2+R_3 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right] \begin{array}{l} \\ \\ -R_3 \end{array}$$

row echelon form

$$x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10$$
$$x_3 + 2x_5 = -3$$
$$x_4 - 4x_5 = 7$$

backsolve

$$x_5 = t_5 \in \mathbb{R}$$
$$x_4 = 7 + 4t_5$$
$$x_3 = -3 - 2t_5$$
$$x_2 = t_2 \in \mathbb{R}$$
$$x_1 = 10 + 2t_2 - 3(-3-2t_5) - 2(7+4t_5) - t_5$$

whew!

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 0 & 9 & -4 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right] \begin{array}{l} -2R_3+R_1 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{array}\right] \begin{array}{l} -3R_2+R_1 \\ \\ \\ \end{array}$$

reduced row echelon form.

$$x_1 - 2x_2 + 3x_5 = 5$$
$$x_3 + 2x_5 = -3$$
$$x_4 - 4x_5 = 7$$

$$x_5 = t_5 \in \mathbb{R}$$
$$x_4 = 7 + 4t_5$$
$$x_3 = -3 - 2t_5$$
$$x_2 = t_2 \in \mathbb{R}$$
$$x_1 = 5 + 2t_2 - 3t_5$$

Maple says:

```
> with(LinearAlgebra) :   # matrix and linear algebra library
> A := Matrix(3, 5, [1,−2, 3, 2, 1,
                     2,−4, 8, 3, 10,
                     3,−6, 10, 6, 5]) :
  b := Vector([10, 7, 27]) :
  ⟨A|b⟩;     # the mathematical augmented matrix doesn't actually have
             # a vertical line between the end of A and the start of b
  ReducedRowEchelonForm(⟨A|b⟩);
```

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix} \quad \textbf{(1)}$$

```
> LinearSolve(A, b);
      #  this command will actually write down the general solution, using
      # Maple's way of writing free parameters, which actually makes
      # some sense.  Generally when there  are free parameters involved,
      #  there will be equivalent ways to express the solution that may
      # look different.  But usually Maple's version will look like yours,
      # because it's using the same algorithm and choosing the free parameters
      # the same way too.
```

$$\begin{bmatrix} 5 + 2\,\_t_2 - 3\,\_t_5 \\ \_t_2 \\ -3 - 2\,\_t_5 \\ 7 + 4\,\_t_5 \\ \_t_5 \end{bmatrix} \quad \textbf{(2)}$$

```
>
```