Friday: mostly doing Wednesday material
I'll post next week's notes by 2:00 p.m.

Math 2250-004
Fri Feb 24
4.1 - 4.3 Concepts related to linear combinations of vectors.

Exercise 1) Vocabulary review (these need to be memorized!)

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is any sum of scalar multiples

i.e. any $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$

The span of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

$\text{span}\{\vec{v}_1\} = \{t\vec{v}_1 \text{ such that } t \in \mathbb{R}\}$
position vectors of a line thru origin

$\text{span}\{\vec{v}_1, \vec{v}_2\} = \{s\vec{v}_1 + t\vec{v}_2, \text{ such that } s, t \in \mathbb{R}\}$
position vectors of a plane thru $\vec{0}$

$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ = all possible linear combo's of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$
$= \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n \text{ with } c_1, c_2, \ldots c_n \in \mathbb{R}\}$

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent iff

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent iff

Covering these notes on Monday 2/27

• Keep recalling that for vectors in $\mathbb{R}^m$ all linear combination questions can be reduced to matrix questions because any linear combination like the one on the left is actually just the matrix product on the right:

$$\underbrace{c_1\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ 0 \\ a_{mn} \end{bmatrix}}_{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n} = \begin{bmatrix} c_1a_{11} + c_2a_{12} + \ldots \\ c_1a_{21} + c_2a_{22} + \ldots \\ \vdots \\ c_1a_{m1} + c_2a_{m2} + \ldots \end{bmatrix} = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \\ \begin{matrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & & \vec{v}_m \end{matrix} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Definition Let W be a subset of $\mathbb{R}^m$ that is closed under addition and scalar multiplication; in other words

α) Whenever $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in W$ then $\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in W$

β) Whenever $\underline{\boldsymbol{v}} \in W$ and $c \in \mathbb{R}$ then $c\underline{\boldsymbol{v}} \in W$.

Then W is called a subspace of $\mathbb{R}^m$.

Notice that the span of any collection of vectors is a subspace because if you add two linear combinations of vectors, the sum is still a linear combination of the (same) vectors; and if you multiply a linear combination by a constant it is still a linear combination.

Definition Let W be a subspace. If $W = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ and if $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent, then we say that they are a basis for W. (The word "basis" makes sense because the entire subspace can be reconstructed by taking linear combinations of the basis vectors, and the linear combinations coefficients for each element in W are unique.

The number of vectors in a (any) basis for a subspace W is called the dimension of W

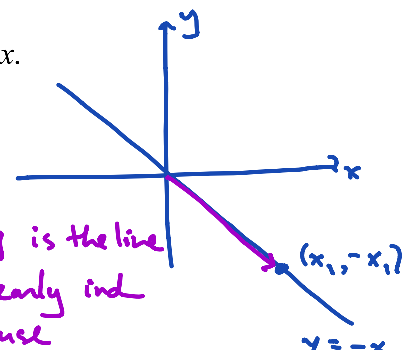
Examples (explain answers!)

Exercise 2:

a) Show $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a basis for the line in $\mathbb{R}^2$ with implicit equation $y = -x$. L

Let $\begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ satisfy

$= x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



• so span$\{\vec{v}_1\}$ is the line

• $\{\vec{v}_1\}$ is linearly ind because

$c_1\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} c_1 \\ -c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow c_1 = 0$

so dimension of L is 1!

b) $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are a basis for $\mathbb{R}^2$.

($\{\vec{v}_1, \vec{v}_2\}$ is a basis)

(The vectors $\underline{\boldsymbol{e}}_1 = \hat{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \underline{\boldsymbol{e}}_2 = \hat{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the so-called "standard basis" for $\mathbb{R}^2$.)

$\dim(\mathbb{R}^2) = 2$!

• $\vec{v}_1, \vec{v}_2$ must span $\mathbb{R}^2$

• $\vec{v}_1, \vec{v}_2$ must be linearly independent

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}$

$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

• yields span (formula for $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ for each $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}$

• yields lin ind set $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

c) $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ are not a basis for $\mathbb{R}^2$.

$\vec{v}_3 = -3.5\vec{v}_1 + 1.5\vec{v}_2$

$\begin{bmatrix} -2 \\ 8 \end{bmatrix} = -3.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

these vectors are linearly dependent

not a basis for $\mathbb{R}^2$

Use matrix theory to explain why

d) Fewer than two vectors cannot be a basis for all of $\mathbb{R}^2$.

a single vector cannot span $\mathbb{R}^2$:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

want

$$c_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left[\begin{array}{c|c} u_1 & b_1 \\ u_2 & b_2 \end{array}\right] \to \left[\begin{array}{c|c} 1 & d_1 \\ 0 & d_2 \end{array}\right]$$

I would $d_2 = 0$ to solve this but a priori, d_2 could be anything.

e) More than two vectors cannot be a basis for all of $\mathbb{R}^2$.

independence must fail.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0} \qquad n>2$$

$$\left[\begin{array}{c|c|c|c|c} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n & 0 \end{array}\right]$$

at least $n-2$ free parameters so lots of solutions $\vec{c}$, besides $c_1 = c_2 = 0$.

f) A choice of exactly two vectors $\{\underline{v}_1, \underline{v}_2\}$ will be a a basis of $\mathbb{R}^2$ if and only if the reduced row echelon form of the matrix having those two vectors as columns is the identity matrix.

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}$$

$$\left[\begin{array}{c|c|c} \vec{v}_1 & \vec{v}_2 & \vec{b} \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & d_1 \\ 0 & 1 & d_2 \end{array}\right]$$

in this case $\vec{v}_1, \vec{v}_2$ are a basis (this computation verifies linear independence & span).

OR

$$\left[\begin{array}{cc|c} 1 & * & d_1 \\ 0 & 0 & d_2 \end{array}\right]$$

neither span nor independence.

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}$

Exercise 3a) $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ are a basis for the plane with implicit equation $2x + y - z = 0$ (See Exercise 2d in Tuesday's notes.)

$\dim(\text{span}\{\vec{v}_1, \vec{v}_2\}) = 2$

$$\left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 2 & b_2 \\ 2 & 0 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 1 & b_2/2 \\ 0 & 0 & \boxed{2b_1 + b_2 - b_3} \end{array}\right]$$

if $2b_1 + b_2 - b_3 = 0$ $\rightarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 + \frac{b_2}{2} \\ 0 & 1 & b_2/2 \\ 0 & 0 & 0 \end{array}\right]$

span! $c_1 = b_1 + b_2/2$, $c_2 = b_2/2$

also shows independence ($\vec{b} = \vec{0} \Rightarrow c_1 = c_2 = 0$)

b) $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$ are not a basis for the plane with implicit equation $2x + y - z = 0$, even though all three vectors lie on the plane. They are also not a basis for $\mathbb{R}^3$.

dependent

$\vec{v}_3$ satisfies $2x + y - z = 0$

$2(-1) + 6 - 4 = 0$ ✓.

> $ReducedRowEchelonForm\left(\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 6 \\ 2 & 0 & 4 \end{array}\right]\right);$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \quad (3)$$

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{v}_3$

$c_1 = 2$, $c_2 = 3$!

$2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$!

c) $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$, $\underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are a basis for $\mathbb{R}^3$.

(The vectors $\underset{\hat{i}}{\underline{e}_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underset{\hat{j}}{\underline{e}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underset{\hat{k}}{\underline{e}_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are the so-called "standard basis" for $\mathbb{R}^3$.)

> $ReducedRowEchelonForm\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}\right);$ $\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} d_1 \\ d_2 \\ d_3 \end{matrix} \quad (4)$$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{b}$

has unique soln

$c_1 = d_1$

$c_2 = d_2$

$c_3 = d_3$

shows "span" & "independence"

Exercise 4) Use properties of reduced row echelon form matrices to answer the following questions:
4a) Why must more than 3 vectors in $\mathbb{R}^3$ always be linearly dependent?
4b) Why can fewer than 3 vectors never span $\mathbb{R}^3$?
(So every basis of $\mathbb{R}^3$ must have exactly three vectors.)
4c) If you are given 3 vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ in $\mathbb{R}^3$, what is the condition on the reduced row echelon form of the 3×3 matrix $\langle \underline{v}_1 | \underline{v}_2 | \underline{v}_3 \rangle$ that guarantees they're linearly independent? That guarantees they span $\mathbb{R}^3$? That guarantees they're a basis of $\mathbb{R}^3$?
4d) What is the dimension of $\mathbb{R}^3$?

4a) $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$n > 3$

at least $n-3$ columns in reduced matrix without leading 1's

at least $n-3$ free parameters

so lots of dependencies

4b) $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}$

$\left[\begin{array}{cc|c} \vec{v}_1 & \vec{v}_2 & \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \vdots & \vdots & \vdots \\ 0 & 0 & d_3 \end{array}\right]$

so span fails

4c) $\left[\begin{array}{ccc|c} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{b} \end{array}\right]$

case I

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{array}\right]$

both "span" & "ind"

case II

$\left[\begin{array}{ccc|c} & & & \vdots \\ 0 & 0 & 0 & d_3 \end{array}\right]$

ind & span fail.

Math 2250-004
Week 8: Finish sections 4.2-4.4 and linear combinations concepts, and then begin Chapter 5 on linear differential equations, sections 5.1-5.2.

Mon Feb 27: Use last Friday's notes to talk about linear independence/dependence, span, subspaces and bases. That discussion will continue into these notes. Here is a summary of vocabulary terms so far:

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ is

any vector $\underline{\boldsymbol{v}}$ *that is a sum of scalar multiples of those vectors, i.e. any* $\underline{\boldsymbol{v}}$ expressible as

$$\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n .$$

The span of $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n\}$ is

the collection of all possible linear combinations:

$$span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n\} := \{\underline{\boldsymbol{v}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n \text{ such that each } c_i \in \mathbb{R}, i = 1, 2, \dots n\}$$

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ are linearly independent means

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n = 0 \Rightarrow c_1 = c_2 = \dots = c_n = 0.$$

(Equivalently, no $\underline{\boldsymbol{v}}_j$ *can be expressed as a linear combination of some of the other vectors in the collection.)*

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ are linearly dependent means

they are not linearly independent.

(Equivalently, some $\underline{\boldsymbol{v}}_j$ *can be expressed as a linear combination of some of the other vectors in the collection.)*

....

In Friday's notes but not yet discussed before Monday:

A subset W of $\mathbb{R}^n$ is a subspace means

that W *is closed under addition and scalar multiplication:*

$$(\alpha) \qquad \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in W \Rightarrow \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in W,$$
$$(\beta) \qquad \underline{\boldsymbol{u}} \in W, c \in \mathbb{R} \Rightarrow c\,\underline{\boldsymbol{u}} \in W.$$

any linear combo of elts of a subspace W is still in the subspace

Let W be a subspace. A basis for W is

a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \dots \underline{\boldsymbol{v}}_n$ *that are linearly independent and span* W.

Equivalently, each $\underline{\boldsymbol{w}} \in W$ can be written as $\underline{\boldsymbol{w}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \dots + c_n\underline{\boldsymbol{v}}_n$ *for exactly one choice of linear combination coefficients* $c_1, c_2, \dots c_n$.

The dimension of a subspace W is the number of vectors in a basis for W . (It turns out that all bases for a subspace always have the same number of vectors.)

Note: Subspaces are special subsets because they are closed with respect to all linear combinations (since linear combinations are built up by successive scalar multiplication and addition operations). The reason why we use the word "subspaces" for these special subsets is because there is a more general notion of vector space for collections of objects that can be added and scalar multiplied so that the usual addition and scalar multiplication axioms hold. $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, .. $\mathbb{R}^m$ are vector spaces. Subspaces of $\mathbb{R}^m$ are also vector spaces in their own right. In Chapter 5 we'll be focusing on function vector spaces (since you can add and scalar multiply functions), and subspaces of those. The purpose will be to understand the solution spaces for higher order linear differential equations, and applications.

- Check that $y_1, y_2, \ldots y_n$ **span** the solution space: Consider any solution $y(x)$ to the DE. We can compute its vector of initial values

$$\begin{bmatrix} y(x_0) \\ y'(x_0) \\ y''(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix} := \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

Now consider a linear combination $z = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$. Compute its initial value vector, and notice that you can write it as the product of the <u>Wronskian matrix</u> at x_0 times the vector of linear combination coefficients:

$$\begin{bmatrix} z(x_0) \\ z'(x_0) \\ \vdots \\ z^{(n-1)}(x_0) \end{bmatrix} = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

We've chosen the $y_1, y_2, \ldots y_n$ so that the Wronskian matrix at x_0 has an inverse, so the matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

has a unique solution $\underline{\mathbf{c}}$. For this choice of linear combination coefficients, the solution $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ has the same initial value vector at x_0 as the solution $y(x)$. By the uniqueness half of the existence-uniqueness theorem, we conclude that

$$y(x) = z(x) = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

Thus $y_1, y_2, \ldots y_n$ <u>span</u> the solution space.

- **<u>linear independence</u>**: If a linear combination $c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \equiv 0$, then differentiate this identity $n - 1$ times, and then substitute $x = x_0$ into the resulting n equations. This yields the Wronskian matrix equation above, with $\left[b_0, b_1, \ldots b_{n-1}\right]^T = [0, 0, \ldots, 0]^T$. So the matrix equation above implies that $\left[c_1, c_2, \ldots c_n\right]^T = \underline{\mathbf{0}}$. So $y_1, y_2, \ldots y_n$ are also <u>linearly independent</u>.

- Thus $y_1, y_2, \ldots y_n$ are a <u>basis</u> for the solution space and the general solution to the homogeneous DE can be written as

$$y_H = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n.$$

$\square$

• In section 5.2 there is a focus on testing whether collections of functions are linearly independent or not. This is important for finding bases for the solution spaces to homogeneous linear DE's because of the fact that if we find n linearly independent solutions to the n^{th} order homogeneous DE, they will automatically span the n-dimensional the solution space. (We discussed this general vector space "magic" fact on Wednesday.) And checking just linear independence is sometimes easier than also checking the spanning property.

<u>Ways to check whether functions $y_1, y_2, \ldots y_n$ are linearly independent on an interval:</u>

In all cases you begin by writing the linear combination equation

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

where "0" is the zero function which equals 0 for all x on our interval.

<u>Method 1)</u> Plug in different $x-$ values to get a system of algebraic equations for $c_1, c_2 \ldots c_n$. Either you'll get enough "different" equations to conclude that $c_1 = c_2 = \ldots = c_n = 0$, or you'll find a likely dependency.

<u>Exercise 3)</u> Use method 1 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1, \ y_2(x) = x, \ y_3(x) = x^2$$

are linearly independent. (These functions show up in the homework due Monday.) For example, try the system you get by plugging in $x = 0, -1, 1$ into the equation

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

Method 2) If your interval stretches to $+\infty$ or to $-\infty$ and your functions grow at different rates, you may be able to take limits (after dividing the dependency equation by appropriate functions of x), to deduce independence.

Exercise 4) Use method 2 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Hint: first divide the dependency equation by the fastest growing function, then let $x \to \infty$.

Method 3) If

$$c_1 y_1 + c_2 y_2 + \ldots + c_n y_n = 0$$

$\forall\ x \in I$, then we can take derivatives to get a system

$$\begin{aligned} c_1 y_1 + c_2 y_2 + \ldots + c_n y_n &= 0 \\ c_1 y_1' + c_2 y_2' + \ldots + c_n y_n' &= 0 \\ c_1 y_1'' + c_2 y_2'' + \ldots + c_n y_n'' &= 0 \\ \\ c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)} + \ldots + c_n y_n^{(n-1)} &= 0 \end{aligned}$$

(We could keep going, but stopping here gives us n equations in n unknowns.)
Plugging in any value of x_0 yields a homogeneous algebraic linear system of n equations in n unknowns, which is equivalent to the Wronskian matrix equation

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y'_n(x_0) \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

If this Wronskian matrix is invertible at even a single point $x_0 \in I$, then the functions are linearly independent! (So if the determinant is <u>NON</u>-zero at even a single point $x_0 \in I$, then the functions are independent....strangely, even if the determinant was zero for all $x \in I$, then it could still be true that the functions are independent....but that won't happen if our n functions are all solutions to the same n^{th} order linear homogeneous DE.)

Exercise 5) Use method 3 for $I = \mathbb{R}$, to show that the functions

$$y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$$

are linearly independent. Use $x_0 = 1$.

Remark 1) Method 3 is usually not the easiest way to prove independence. But we and the text like it when studying differential equations because as we've seen, the Wronskian matrix shows up when you're trying to solve initial value problems using

$$y = y_P + y_H = y_P + c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$$

as the general solution to

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f.$$

This is because, if the initial conditions for this inhomogeneous DE are

$$y(x_0) = b_0, y'(x_0) = b_1, \ldots y^{(n-1)}(x_0) = b_{n-1}$$

then you need to solve matrix algebra problem

$$\begin{bmatrix} y_P(x_0) \\ y_P'(x_0) \\ \vdots \\ y_P^{(n-1)}(x_0) \end{bmatrix} + \begin{bmatrix} y_1(x_0) & y_2(x_0) & \ldots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \ldots & y'_n(x_0) \\ \vdots & \vdots & \ldots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \ldots & y_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{bmatrix}.$$

for the vector $[c_1, c_2, \ldots, c_n]^T$ of linear combination coefficients. And so if you're using the Wronskian matrix method, and the matrix is invertible at x_0 then you are effectively directly checking that $y_1, y_2, \ldots y_n$ are a basis for the homogeneous solution space, and because you've found the Wronskian matrix you are ready to solve any initial value problem you want by solving for the linear combination coefficients above.

Remark 2) There is a seemingly magic consequence in the situation above, in which $y_1, y_2, \ldots y_n$ are all solutions to the same n^{th}-order homogeneous DE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

(even if the coefficients aren't constants): If the Wronskian matrix of your solutions $y_1, y_2, \ldots y_n$ is invertible at a single point x_0, then $y_1, y_2, \ldots y_n$ are a basis because linear combinations uniquely solve all IVP's at x_0. But since they're a basis, that also means that linear combinations of $y_1, y_2, \ldots y_n$ solve all IVP's at any other point x_1. This is only possible if the Wronskian matrix at x_1 also reduces to the identity matrix at x_1 and so is invertible there too. In other words, the Wronskian determinant will either be non-zero $\forall\ x \in I$, or zero $\forall\ x \in I$, when your functions $y_1, y_2, \ldots y_n$ all happen to be solutions to the same n^{th} order homogeneous linear DE as above.

Exercise 6) Verify that $y_1(x) = 1,\ y_2(x) = x,\ y_3(x) = x^2$ all solve the third order linear homogeneous DE

$$y''' = 0,$$

and that their Wronskian determinant is indeed non-zero $\forall\ x \in \mathbb{R}$.