Math 2250-004
Tue Feb 14

- yesterday: how to find inverse matrices
- today: "determinants"
  magic formula for A^{-1} using determinants.
- HW due Thurs @ start of lab

Continue with section 3.6 Determinants

2/21 start here systematically

The effective way to compute determinants for larger-sized matrices without lots of zeroes is to not use the definition, but rather to use the following facts, which track how elementary row operations affect determinants:

- (1a) Swapping any two rows changes the sign of the determinant.

  e.g. swap rows 1 & 3

  $$\begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ d & e & f \end{vmatrix} \text{ vs } \begin{vmatrix} d & e & f \\ \alpha & \beta & \gamma \\ a & b & c \end{vmatrix}$$

  proof: This is clear for 2×2 matrices, since

  $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad \ . = -(ad-bc).$$

  $$-\alpha\begin{vmatrix} b & c \\ e & f \end{vmatrix} + \beta\begin{vmatrix} a & c \\ d & f \end{vmatrix} - \gamma\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

  $$-\alpha\begin{vmatrix} e & f \\ b & c \end{vmatrix} + \beta\begin{vmatrix} d & f \\ a & c \end{vmatrix} - \gamma\begin{vmatrix} d & e \\ a & b \end{vmatrix}$$

  = opposite of red determinant

  For 3×3 determinants, expand across the row *not* being swapped, and use the 2×2 swap property to deduce the result. Prove the general result by induction: once it's true for $n \times n$ matrices you can prove it for any $(n+1) \times (n+1)$ matrix, by expanding across a row that wasn't swapped, and applying the $n \times n$ result.

  (1b) Thus, if two rows in a matrix are the same, the determinant of the matrix must be zero: on the one hand, swapping those two rows leaves the matrix and its determinant unchanged; on the other hand, by (1a) the determinant changes its sign. The only way this is possible is if the determinant is zero.   swap: $-|A| = |A| \Rightarrow 0 = 2|A| \Rightarrow |A| = 0$

- (2a) If you factor a constant out of a row, then you factor the same constant out of the determinant. Precisely, using $\mathcal{R}_i$ for i^{th} row of A, and writing $\mathcal{R}_i = c\,\mathcal{R}_i^*$

  e.g. $\begin{vmatrix} 1 & 2 \\ 6 & 6 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$
  $-6 \overset{?}{=} 6(-1)$ ✓

  $$|A| = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i \\ \mathcal{R}_n \end{vmatrix} = \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ c\,\mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix} = c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \vdots \\ \mathcal{R}_i^* \\ \mathcal{R}_n \end{vmatrix}.$$

  proof: expand across the i^{th} row, noting that the corresponding cofactors don't change, since they're computed by deleting the i^{th} row to get the corresponding minors:

  $$det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} (c\, a_{ij}^*) C_{ij} = c \sum_{j=1}^{n} a_{ij}^* C_{ij} = c\, det(A^*) \ .$$

- (2b) Combining (2a) with (1b), we see that if one row in A is a scalar multiple of another, then $det(A) = 0$.

- (3) If you replace row i of A by its sum with a multiple of another row, then the determinant is unchanged! Expand across the i^{th} row:

$$\begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_i + c\,\mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = \sum_{j=1}^{n} \left(a_{ij} + c\,a_{kj}\right) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + c \underbrace{\sum_{j=1}^{n} a_{kj} C_{ij}}_{} = det(A) + c \begin{vmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_k \\ \mathcal{R}_k \\ \mathcal{R}_n \end{vmatrix} = det(A) + 0\ .$$

$a_{ij}C_{ij} + c\,a_{kj}C_{ij}$

i^{th} row

Remark: The analogous properties hold for corresponding "elementary column operations". In fact, the proofs are almost identical, except you use column expansions.

Exercise 1) Recompute $\begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$ $= 15.$ from yesterday (using row and column expansions we always got an answer of 15 then.) This time use elementary row operations (and/or elementary column operations).

$$= \begin{array}{r} \\ \\ -2R_1+R_3 \end{array} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -6 & 3 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= -3 \begin{array}{r} \\ -R_3+R_2 \\ \end{array} \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= -3 \begin{array}{r} \\ \\ -2R_2+R_3 \end{array} \underbrace{\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{vmatrix}} = (-3)(1\cdot 1\cdot(-5)) = 15 \checkmark$$

$$\begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = 18-12 = 6 \checkmark$$

$$\| $$

$$2\cdot 3 \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 2\cdot 3\cdot 1$$

Exercise 2) Compute $\begin{vmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{vmatrix}$. $= -0 + 1 \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -1 & -2 & 1 \end{vmatrix} - 0 + 0$

$= \ldots$

OR

$$\begin{array}{r} \\ -2R_1+R_2 \\ -2R_1+R_3 \\ R_1+R_4 \end{array} \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -3 & 3 \end{vmatrix} = \begin{array}{r} \\ \\ \\ R_3+R_4 \end{array} \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\| \qquad\qquad \|$$

$$0 \qquad\qquad 0$$

Maple check:

```
> with(LinearAlgebra) :
> A := Matrix(4, 4, [1, 0, -1, 2, 2, 1, 1, 0, 2, 0, 1, 1, -1, 0, -2, 1]);
  Determinant(A);
```

$$A := \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ -1 & 0 & -2 & 1 \end{bmatrix}$$

$$0 \qquad \textbf{(1)}$$

know : A^{-1} exists
$\Longleftrightarrow$
$rref(A) = I$
$\Longleftrightarrow$ $|A| \neq 0$

$\boxed{\text{\underline{Theorem:} Let } A_{n \times n}. \text{ Then } A^{-1} \text{ exists if and only if } det(A) \neq 0.}$

proof: We already know that A^{-1} exists if and only if the reduced row echelon form of A is the identity matrix. Now, consider reducing A to its reduced row echelon form, and keep track of how the determinants of the corresponding matrices change: As we do elementary row operations,

- if we swap rows, the sign of the determinant switches.
- if we factor non-zero factors out of rows, we factor the same factors out of the determinants.
- if we replace a row by its sum with a multiple of another row, the determinant is unchanged.

Thus,

$$|A| = c_1|A_1| = c_1c_2|A_2| = \ldots = \underbrace{c_1c_2 \ldots c_N}_{\neq 0}|rref(A)|$$

where the nonzero c_k's arise from the three types of elementary row operations. If $rref(A) = I$ its determinant is 1, and $|A| = c_1c_2 \ldots c_N \neq 0$. If $rref(A) \neq I$ then its bottom row is all zeroes and its determinant is zero, so $|A| = c_1c_2 \ldots c_N(0) = 0$. Thus $|A| \neq 0$ if and only if $rref(A) = I$ if and only if A^{-1} exists.

<u>Remark:</u> Using the same ideas as above, you can show that $det(AB) = det(A)det(B)$. This is an important identity that gets used, for example, in multivariable change of variables formulas for integration, using the Jacobian matrix. (It is <u>not</u> true that $\cancel{det(A + B) = det(A) + det(B)}$.) Here's how to show $det(AB) = det(A)det(B)$: The key point is that if you do an elementary row operation to AB, that's the same as doing the elementary row operation to A, and then multiplying by B. With that in mind, if you do exactly the same elementary row operations as you did for A in the theorem above, you get

$$|AB| = c_1|A_1B| = c_1c_2|A_2B| = \ldots = c_1c_2 \ldots c_N|rref(A)B|.$$

If $rref(A) = I$, then from the theorem above, $|A| = c_1c_2 \ldots c_N$, and we deduce $|AB| = |A||B|$. If $rref(A) \neq I$, then its bottom row is zeroes, and so is the bottom row of $rref(A)B$. Thus $|AB| = 0$ and also $|A||B| = 0$.

(read if interested)

There is a "magic" formula for the inverse of square matrices A (called the "adjoint formula") that uses the determinant of A along with the cofactor matrix of A.
In order to understand the $n \times n$ magic formula for matrix inverses, we first need to talk about matrix *transposes:*

<u>Definition:</u> Let $B_{m \times n} = \left[b_{i\,j}\right]$. Then the <u>transpose</u> of B, denoted by B^T is an $n \times m$ matrix defined by

$$entry_{i\,j}\left(B^T\right) := entry_{j\,i}(B) = b_{j\,i}\,.$$

The effect of this definition is to turn the columns of B into the rows of B^T :

$$entry_i\left(col_j(B)\right) = b_{i\,j}\,.$$

$$entry_i\left(row_j\left(B^T\right) = entry_{j\,i}\left(B^T\right) = b_{i\,j}\,.$$

And to turn the rows of B into the columns of B^T:

$$entry_j\left(row_i(B)\right) = b_{i\,j}$$

$$entry_j\left(col_i\left(B^T\right)\right) = entry_{j\,i}\left(B^T\right) = b_{i\,j}\,.$$

<u>Exercise 3)</u> explore these properties with the identity

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = B_{2\times 3} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}. = B^T_{3\times 2}$$

e.g. $\quad [1, 2, 3, 4]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

Theorem: Let $A_{n \times n}$, and denote its cofactor matrix by $cof(A) = [C_{ij}]$, with $C_{ij} = (-1)^{i+j} M_{ij}$, and $M_{ij} =$ the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.
Define the adjoint matrix to be the transpose of the cofactor matrix:

$$Adj(A) := cof(A)^T$$

Then, when A^{-1} exists it is given by the formula

$$\boxed{A^{-1} = \frac{1}{det(A)} Adj(A)\ .}$$

Exercise 4) Show that in the 2×2 case this reproduces the formula

$$\left.\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.\right\}$$

know this

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$Adj(A) = (cof(A))^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$cof(A) = \begin{bmatrix} +d & -c \\ -b & a \end{bmatrix}.$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \checkmark$$

Exercise 5) For our friend $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ we worked out $cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix}$ and $det(A) = 15$.

Use the Theorem to find A^{-1} and check your work. Does the matrix multiplication relate to the dot products we computed between various rows of A and rows of $cof(A)$?

$$Adj(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} Adj(A)$$

$$= \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \frac{1}{15}\begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Exercise 6) Continuing with our example,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix} \qquad cof(A) = \begin{bmatrix} 5 & 2 & -6 \\ 0 & 3 & 6 \\ 5 & -1 & 3 \end{bmatrix} \qquad Adj(A) = \begin{bmatrix} 5 & 0 & 5 \\ 2 & 3 & -1 \\ -6 & 6 & 3 \end{bmatrix}$$

$A^{-1} = \frac{1}{|A|} Adj(A)$

$(A)\frac{1}{|A|} Adj(A) = I$

6a) The $(1, 1)$ entry of $(A)(Adj(A))$ is $15 = 1 \cdot 5 + 2 \cdot 2 + (-1)(-6)$. Explain why this is $det(A)$, expanded across the first row.

$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$A\, Adj(A) = |A|\, I$

6b) The $(2, 1)$ entry of $(A)(Adj(A))$ is $0 \cdot 5 + 3 \cdot 2 + (1)(-6) = 0$. Notice that you're using the same cofactors as in (4a). What matrix, which is obtained from A by ~~keeping two of~~ the rows, but replacing a third one with one of those two, is this the determinant of?

this is a +.

$$= \begin{vmatrix} 0 & 3 & 1 \\ 0 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 0 \cdot \underset{5}{C_{11}} + 3\, C_{12} + 1\, C_{13}$$

6c) The $(3, 2)$ entry of $(A)(Adj(A))$ is $2 \cdot 0 - 2 \cdot 3 + 1 \cdot 6 = 0$. What matrix (which uses two rows of A) is this the determinant of?

$= 0$ because 2 rows are the same.

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{vmatrix} = 0$$

$$[A][Adj(A)] = \begin{bmatrix} |A| & & 0 \\ & |A| & \\ 0 & & \ddots \; |A| \end{bmatrix}$$

$$A\left(\frac{1}{|A|} Adj(A)\right) = \cancel{|A|}\, I$$

If you completely understand 6abc, then you have realized why

$$[A][Adj(A)] = det(A)[I]$$

for every square matrix, and so also why

$$A^{-1} = \frac{1}{det(A)} Adj(A) .$$

Precisely,

$$entry_{i\,i}\, A(Adj(A)) = row_i(A) \cdot col_i(Adj(A)) = row_i(A) \cdot row_i(cof(A)) = det(A),$$

expanded across the i^{th} row.

On the other hand, for $i \neq k$,

$$entry_{k\,i}\, A(Adj(A)) = row_k(A) \cdot col_i(Adj(A)) = row_k(A) \cdot row_i(cof(A)) .$$

This last dot produce is zero because it is the determinant of a matrix made from A by replacing the i^{th} row with the k^{th} row, expanding across the i^{th} row, and whenever two rows are equal, the determinant of a matrix is zero.

There's a related formula for solving for individual components of $\underline{\boldsymbol{x}}$ when $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ has a unique solution ($\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$). This can be useful if you only need one or two components of the solution vector, rather than all of it:

Cramer's Rule: Let $\underline{\boldsymbol{x}}$ solve $A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, for invertible A. Then

$$x_k = \frac{det(A_k)}{det(A)}$$

where A_k is the matrix obtained from A by replacing the k^{th} column with $\underline{\boldsymbol{b}}$.

proof: Since $\underline{\boldsymbol{x}} = A^{-1}\underline{\boldsymbol{b}}$ the k^{th} component is given by

$$\begin{aligned} x_k &= entry_k\left(A^{-1}\underline{\boldsymbol{b}}\right) \\ &= entry_k\left(\frac{1}{|A|}Adj(A)\underline{\boldsymbol{b}}\right) \\ &= \frac{1}{|A|}row_k(Adj(A)) \bullet \underline{\boldsymbol{b}} \\ &= \frac{1}{|A|}col_k(cof(A)) \bullet \underline{\boldsymbol{b}}. \end{aligned}$$

Notice that $col_k(cof(A)) \bullet \underline{\boldsymbol{b}}$ is the determinant of the matrix obtained from A by replacing the k^{th} column by $\underline{\boldsymbol{b}}$, where we've computed that determinant by expanding down the k^{th} column! This proves the result.

Exercise 7) Solve $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

7a) With Cramer's rule

7b) With A^{-1}, using the adjoint formula.

$$x = \frac{\begin{vmatrix} 7 & -1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix}} = \frac{9}{9} = 1.$$

$$y = \frac{\begin{vmatrix} 5 & 7 \\ 4 & 2 \end{vmatrix}}{9} = \frac{-18}{9} = -2$$

check. $\begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$ ✓.

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

• Hw due Thursday.
(determinants plus intro to Chptr 4)
• Exams will be returned Friday.

Math 2250-004
Week 7 notes: Sections 4.1-4.3 vector space concepts
Tues Feb 21 Today Tuesday

Wed 2/22 start here

• Finish section 3.6 on Determinants and connections to matrix inverses. Use last week's notes. Then if we have time on Tuesday, begin: Exercise 1.

4.1-4.3 The vector space $\mathbb{R}^m$ and its subspaces; concepts related to linear combinations of vectors.

sums of scalar multiples of vectors

We never wrote it down carefully in Chapter 3, but for any natural number $m = 1, 2, 3 \ldots$ the space $\mathbb{R}^m$ may be thought of in two equivalent ways. In both cases, $\mathbb{R}^m$ consists of all possible $m - tuples$ of numbers:

(i) We can think of those $m - tuples$ as representing points, as we're used to doing for $m = 1, 2, 3$. In this case we can write

$\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4, \ldots$

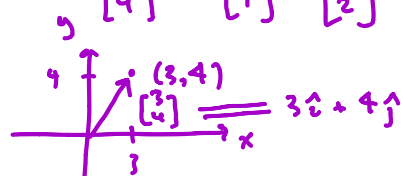
$$\mathbb{R}^m = \{ (x_1, x_2, \ldots, x_m), \ s.t.\ x_1, x_2, \ldots, x_m \in \mathbb{R} \}.$$

$(3,4) \in \mathbb{R}^2, \quad (-1,2,6) \in \mathbb{R}^3$

(ii) We can think of those $m - tuples$ as representing vectors that we can add and scalar multiply. In this case we can write

$$\mathbb{R}^m = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_m \end{bmatrix}, \ s.t.\ x_1, x_2, \ldots, x_m \in \mathbb{R} \right\}.$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



Since algebraic vectors (as above) can be used to measure geometric displacement, one can identify the two models of $\mathbb{R}^m$ as sets by identifying each point $(x_1, x_2, \ldots x_m)$ in the first model with the displacement vector $\underline{\boldsymbol{x}} = [x_1, x_2, \ldots x_m]^T$ from the origin to that point, in the second model, i.e. the position vector. (Notice we just used a transpose, writing a column vector as a transpose of a row vector.)

One of the key themes of Chapter 4 is the idea of linear combinations. These have an algebraic definition (that we've seen before in Chapter 3 and repeat here), as well as a geometric interpretation as combinations of displacements, as we will review in our first few exercises.

Definition: If we have a collection of n vectors $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n\}$ in $\mathbb{R}^m$, then any vector $\underline{\boldsymbol{v}} \in \mathbb{R}^m$ that can be expressed as a sum of scalar multiples of these vectors is called a linear combination of them. In other words, if we can write

$$\underline{\boldsymbol{v}} = c_1 \underline{\boldsymbol{v}}_1 + c_2 \underline{\boldsymbol{v}}_2 + \ldots + c_n \underline{\boldsymbol{v}}_n,$$

then $\underline{\boldsymbol{v}}$ is a linear combination of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$. The scalars $c_1, c_2, \ldots, c_n$ are called the linear combination coefficients.

Example You've probably seen linear combinations in previous math/physics classes. For example you might have expressed the position vector $\underline{\mathbf{r}}$ as a linear combination

$$\vec{\underline{\mathbf{r}}} = x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}}$$

where $\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$ represent the unit displacements in the x,y,z directions, respectively. Since we can express these displacements using Math 2250 notation as

$$\underline{\mathbf{i}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \underline{\mathbf{j}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \underline{\mathbf{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we have

$$\underbrace{x\,\underline{\mathbf{i}} + y\,\underline{\mathbf{j}} + z\,\underline{\boldsymbol{k}}}_{\substack{2210\\1320}} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{2250}.$$

Remarks: When we had free parameters in our explicit solutions to linear systems of equations $A\,\underline{\mathbf{x}} = \underline{\boldsymbol{b}}$ back in Chapter 3, we sometimes rewrote the explicit solutions using linear combinations, where the scalars were the free parameters (which we often labeled with letters that were t, t_4, t_3 etc., rather than with "c's"). When we return to differential equations in Chapter 5 -studying higher order differential equations - then the explicit solutions will also be expressed using "linear combinations", just as we did in Chapters 1 -2, where we used the letter "C" for the single free parameter in first order differential equation solutions:

Definition: If we have a collection $\{y_1, y_2, \ldots, y_n\}$ of n functions $y(x)$ defined on a common interval I, then any function that can be expressed as a sum of scalar multiples of these functions is called a linear combination of them. In other words, if we can write

Chapter 5

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n\,,$$

then y is a linear combination of $y_1, y_2, \ldots, y_n$.

The reason that the same words are used to describe what look like two quite different settings, is that there is a common fabric of mathematics (called vector space theory) that underlies both situations. We shall be exploring these concepts over the next several lectures, using a lot of the matrix algebra theory we've just developed in Chapter 3. This vector space theory will tie in directly to our study of differential equations, in Chapter 5 and subsequent chapters.

Exercise 1) (Linear combinations in $\mathbb{R}^2$... this will also review the geometric meaning of vector addition and scalar multiplication in terms of net displacements.)

Can you get to the point $(-2, 8) \in \mathbb{R}^2$, from the origin $(0, 0)$, by moving only in the $(\pm)$ directions of $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$? Algebraically, this means we want to solve the linear combination problem

$$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

1a) Plot the points $(1, -1)$ and $(1, 3)$, which have position vectors $\underline{\boldsymbol{v}}_1 = [1, -1]^T$ and $\underline{\boldsymbol{v}}_2 = [1, 3]^T$. Compute $\underline{\boldsymbol{v}}_1 + \underline{\boldsymbol{v}}_2$, $\underline{\boldsymbol{v}}_1 - \underline{\boldsymbol{v}}_2$. Plot the points for which these are the position vectors. Plot the line of points having position vectors

$$\{\underline{\boldsymbol{v}}_1 + t\,\underline{\boldsymbol{v}}_2, t \in \mathbb{R}\}.$$

Note: Depending on where you took multivariable calculus you may have written this parametric line in various ways:

$$x = 1 + t$$
$$y = -1 + 3\,t$$

OR

$$\langle x, y \rangle = (1 + t)\,\underline{\mathbf{i}} + (-1 + 3\,t)\,\underline{\mathbf{j}}.$$

1b) Superimpose a grid related to the displacement vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ onto the graph paper below, and, recalling that vector addition yields net displacement, and scalar multiplication yields scaled displacement, try to approximately solve the linear combination problem above, geometrically.

1c) Rewrite the linear combination problem as a matrix equation, and solve it exactly, algebraically.

$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ pt = $(-2, 8)$

$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\vec{v}_1 - \vec{v}_2 = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$

$\{\vec{v}_1 + t\vec{v}_2 \text{ s.t. } t \in \mathbb{R}\}$ is parametric line

$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\} \leftarrow$

OR

$x = 1 + t$
$y = -1 + 3t$

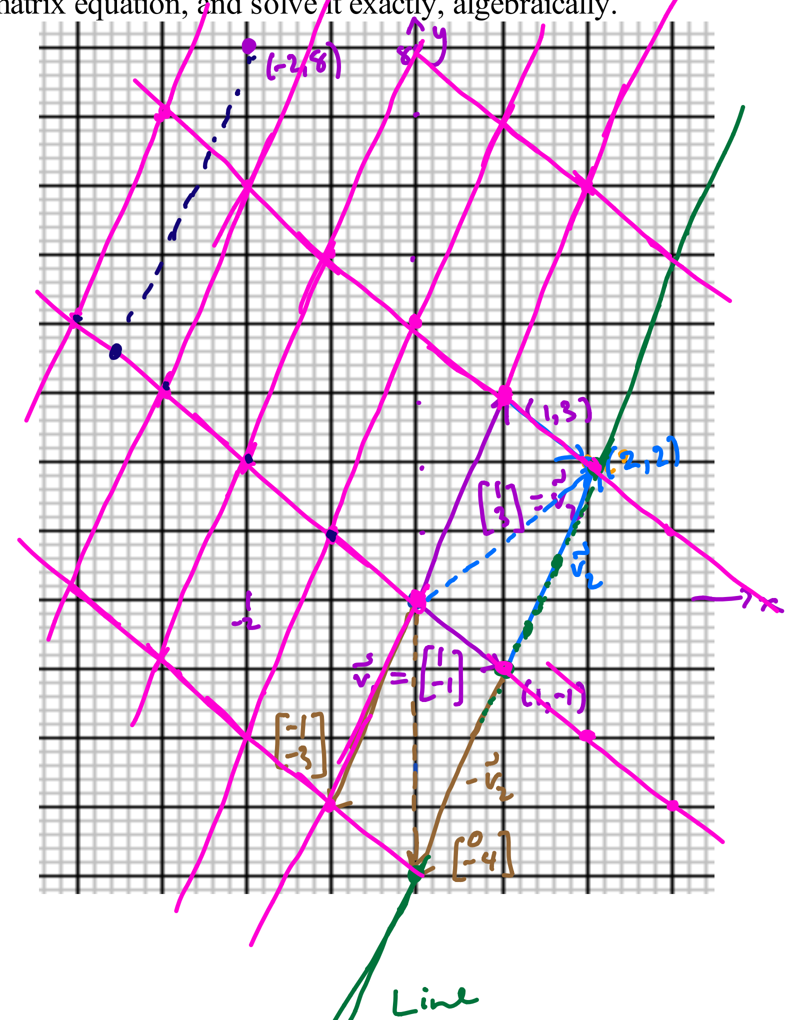
OR

$\vec{r}(t) = (1+t)\hat{i} + (-1+3t)\hat{j}$

guess $-3.5\vec{v}_1 + (1.5)\vec{v}_2 \approx \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

Algebra: $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$

$c_1 + c_2 = -2$
$-c_1 + 3c_2 = 8$



$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$\vec{v}_1 \quad \vec{v}_2$

$\vec{c} = A^{-1}\vec{b}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ +1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 8 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -14 \\ 6 \end{bmatrix} = \begin{bmatrix} -3.5 \\ 1.5 \end{bmatrix}$$

1c) Can you get to any point (x, y) in $\mathbb{R}^2$, starting at $(0, 0)$ and moving only in directions parallel to $\underline{v}_1, \underline{v}_2$? Argue geometrically and algebraically. How many ways are there to express $[x, y]^T$ as a linear combination of $\underline{v}_1$ and $\underline{v}_2$?

Yes

to get to $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ using $c_1\vec{v}_1 + c_2\vec{v}_2$ yields

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

unique soltns c_1, c_2 since A^{-1} exists.

Definition: The span of a collection of vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ in $\mathbb{R}^m$ is the collection of all vectors $\underline{w}$ which can be expressed as linear combinations of $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$. We denote this collection as

$$span\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n\} := \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n \text{ such that each } c_i \in \mathbb{R}\}$$

Remark: The mathematical meaning of the word span is related to the English meaning - as in "wing span" or "span of a bridge", but it's also different. The span of a collection of vectors goes on and on and does not "stop" at the vector or associated endpoint:

Example 1)

$$= \left\{ c\begin{bmatrix} 1 \\ -1 \end{bmatrix}, c \in \mathbb{R} \right\}$$

- In Exercise 1, consider the $span\{\underline{v}_1\} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. This is the set of all vectors of the form

$\begin{bmatrix} c \\ -c \end{bmatrix}$ with free parameter $c \in \mathbb{R}$. This is a line through the origin of $\mathbb{R}^2$ described parametrically, that we're more used to describing with implicit equation $y = -x$ (which is short for $\{(x, y) \in \mathbb{R}^2 \ s.t.\ y = -x\}$). (More precisely, $span\{\underline{v}_1\}$ is the collection of all position vectors for that line.
)

Example 2:

- In Exercise 1 we showed that the span of $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is all of $\mathbb{R}^2$.

Exercise 2) Consider the two vectors $\underline{\boldsymbol{v}}_1 = [1, 0, 2]^T$, $\underline{\boldsymbol{v}}_2 = [-1, 2, 0]^T \in \mathbb{R}^3$.

2a) Sketch these two vectors as position vectors in $\mathbb{R}^3$, using the axes below.

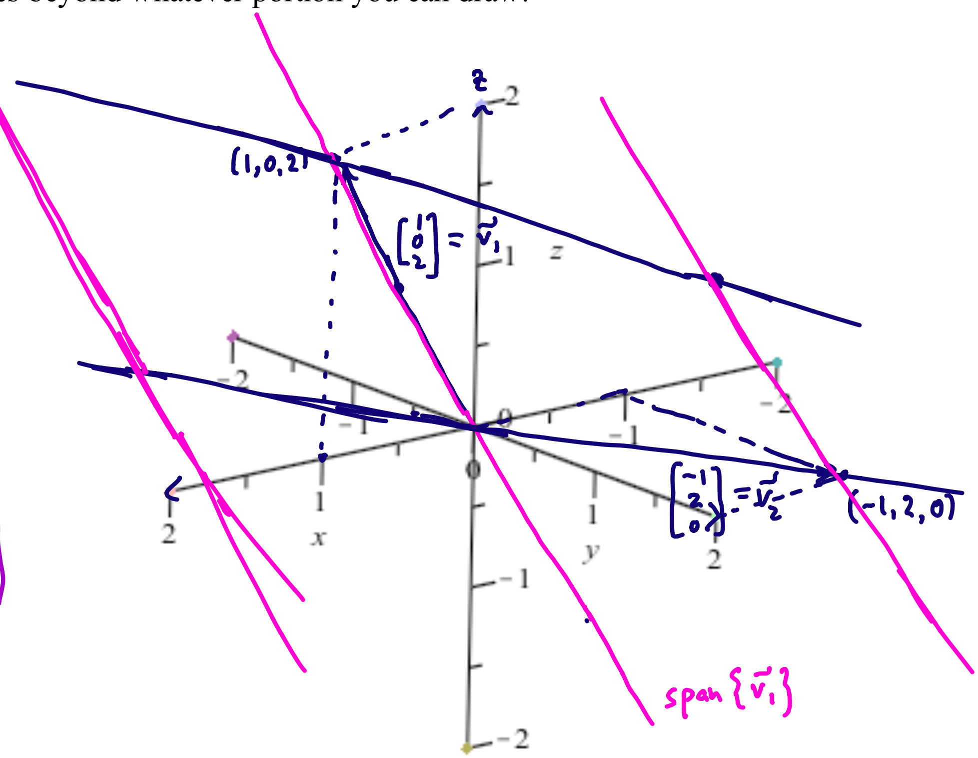
2b) What geometric object is $span\{\underline{\boldsymbol{v}}_1\}$? Sketch a portion of this object onto your picture below.
Remember though, the "span" continues beyond whatever portion you can draw.

2c) What geometric object is $span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$? Sketch a portion of this object onto your picture below.
Remember though, the "span" continues beyond whatever portion you can draw.

$\{t\vec{v}_1,\ t \in \mathbb{R}\}$
line

plane
$\{t\vec{v}_1 + s\vec{v}_2\}$

$\vec{v}_1 = \begin{bmatrix}1\\0\\2\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$



2d) What implicit equation must vectors $[x, y, z]^T$ satisfy in order to be in $span\{\underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2\}$? Hint: For what $[x, y, z]^T$ can you solve the system

$$\underbrace{c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}_{c_1\vec{v}_1} + \underbrace{c_2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{+\, c_2\vec{v}_2} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

for c_1, c_2? Write this an augmented matrix problem and use row operations to reduce it, to see when you get a consistent system for c_1, c_2.

$$\begin{aligned} c_1 - c_2 &= b_1 \\ 2c_2 &= b_2 \\ 2c_1 \quad &= b_3 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 2 & b_2 \\ 2 & 0 & b_3 \end{array}\right]$$

$$\begin{array}{l} \\ R_2/2 \\ -2R_1+R_3 \end{array} \left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 1 & b_2/2 \\ 0 & 2 & -2b_1+b_3 \end{array}\right]$$

$$\begin{array}{l} \\ \\ -2R_2+R_3 \end{array} \left[\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 1 & b_2/2 \\ 0 & 0 & -b_2-2b_1+b_3 \end{array}\right]$$

$$\begin{array}{l} R_2+R_1 \\ \\ \\ \end{array} \left[\begin{array}{cc|c} 1 & 0 & b_1+\frac{b_2}{2} \\ 0 & 1 & b_2/2 \\ 0 & 0 & 0 \end{array}\right] \leftarrow \vec{b} \text{ satisfies}$$

need. $-2b_1 - b_2 + b_3 = 0$

$$\boxed{-2x - y + z = 0}$$

implicit equation of the plane spanned by $\vec{v}_1, \vec{v}_2$

$c_1 = b_1 + \frac{b_2}{2}$
$c_2 = b_2/2$

for $\vec{b} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$ satisfies $-2(2) - 4 + 8 = 0$ ✓.

$c_1 = 2 + \frac{4}{2} = 4$
$c_2 = \frac{4}{2} = 2$

so $\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$ ✓

• §3.6 HW due tomorrow
• §3.6 quiz today.
• if you have Wed time, look at postponed HW before lab tomorrow

Wednesday Feb. 22

We've been talking about "linear combinations" of vectors. Start 8 Finish Tuesday's notes and then continue the discussion here.

covering this on Friday.

Exercise 1:
a) What is the definition of "a linear combination" of the vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$ "

b) What is the "span" of the vectors $\underline{v}_1, \underline{v}_2, \ldots \underline{v}_n$?

Yesterday we interpreted linear combinations geometrically. And, we noticed that to answer natural questions we ended up using matrix theory from Chapter 3 This is because

Exercise 2) By carefully expanding the linear combination below, check than in $\mathbb{R}^m$, the linear combination

$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n$

$$c_1 \begin{bmatrix} a_{11} c_1 \\ a_{21} c_1 \\ \vdots \\ a_{m1} c_1 \end{bmatrix} + c_2 \begin{bmatrix} a_{12} c_2 \\ a_{22} c_2 \\ \vdots \\ a_{m2} c_2 \end{bmatrix} + \ldots + c_n \begin{bmatrix} a_{1n} c_n \\ a_{2n} c_n \\ \vdots \\ a_{mn} c_n \end{bmatrix} = \begin{bmatrix} a_{11} c_1 + a_{12} c_2 + \cdots + a_{1n} c_n \\ a_{21} c_1 + a_{22} c_2 + \cdots + a_{2n} c_n \\ \vdots \\ a_{m1} c_1 + a_{m2} c_2 + \cdots + a_{mn} c_n \end{bmatrix}$$

is always just the matrix times vector product

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_n$

Thus linear combination problems in $\mathbb{R}^m$ can usually be answered using the linear system and matrix techniques we've just been studying in Chapter 3. This will be the main theme of Chapter 4. We've seen this theme in action, in exercises 1,2 in Tuesday's notes.

When we are discussing the span of a collection of vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ we would like to know that we are being efficient in describing this collection, and not wasting any free parameters because of redundancies. This has to do with the concept of "linear independence":

Definition:
a) The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent if no one of the vectors is a linear combination of (some) of the other vectors. The logically equivalent concise way to say this is that the only way $\underline{\mathbf{0}}$ can be expressed as a linear combination of these vectors,

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}},$$

is for all the linear combination coefficients $c_1 = c_2 = \ldots = c_n = 0$.

(or, no $\vec{v}_j$ is a linear combo of some of the other $\vec{v}_i$'s)

b) $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent if at least one of these vectors *is* a linear combination of (some) of the other vectors. The concise way to say this is that there is some way to write $\underline{\mathbf{0}}$ as a linear combination of these vectors

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots + c_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}}$$

where not all of the $c_j = 0$. (We call such an equation a linear dependency. Note that if we have any such linear dependency, then any $\underline{\boldsymbol{v}}_j$ with $c_j \neq 0$ is a linear combination of the remaining $\underline{\boldsymbol{v}}_k$ with $k \neq j$. We say that such a $\underline{\boldsymbol{v}}_j$ is linearly dependent on the remaining $\underline{\boldsymbol{v}}_k$.)

if $\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2$
then $\vec{0} = 2\vec{v}_1 - \vec{v}_2 - \vec{v}_3$

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$
if they're dependent, one is a scalar multiple of the other

Note: Two non-zero vectors are linearly independent precisely when they are not multiples of each other. For more than two vectors the situation is more complicated.

Example (Refer to Exercise 1 Tuesday):
The vectors $\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$ in $\mathbb{R}^2$ are linearly dependent because, as we showed on Tuesday and as we can quickly recheck,

$$\checkmark \begin{bmatrix} -2 \\ 8 \end{bmatrix} = -3.5\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1.5\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}.$$

I can find one that is a linear combo of the others.

We can also write this linear dependency as

$$-3.5\underline{\boldsymbol{v}}_1 + 1.5\underline{\boldsymbol{v}}_2 - \underline{\boldsymbol{v}}_3 = \underline{\mathbf{0}}$$

(or any non-zero multiple of that equation.)

some linear combo adds up to zero, where not all c_j's $= 0$

Exercise 3) Are the vectors $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ linearly independent? How about $\underline{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\underline{v}_3 = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$?

ind. because not scalar multiples!

long way: $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ try to show $c_1 = c_2 = 0$

$$\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$\underbrace{\vec{v}_1, \vec{v}_3}$

yes, not scalar mults.

Exercise 4) For linearly independent vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$, every vector $\underline{\boldsymbol{v}}$ in their span can be written as $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + \ldots + d_n\underline{\boldsymbol{v}}_n$ uniquely, i.e. for exactly one choice of linear combination coefficients $d_1, d_2, \ldots d_n$. This is not true if vectors are dependent. Explain these facts. (You can illustrate these facts with the vectors in Exercise 3.)

$E_1 \quad \vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 + \ldots + d_n\vec{v}_n$

$E_2 \quad \vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + \ldots + b_n\vec{v}_n$

$E_1 - E_2: \quad \vec{0} = (d_1-b_1)\vec{v}_1 + (d_2-b_2)\vec{v}_2 + \ldots + (d_n-b_n)\vec{v}_n$

$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$

if $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ are independent

all the $c_1 = c_2 = \ldots = c_n = 0$

so $b_1 = d_1,\ b_2 = d_2 \ldots b_n = d_n$.

Exercise 5) (Refer to Exercise 2 in Tuesday's notes):

5a) Are the vectors

$$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \underline{\boldsymbol{v}}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

linearly independent? Yes. Not scalar multiples !!!

(this only works for two vectors).

5b) Show that the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$$

are linearly dependent (even though no two of them are scalar multiples of each other). What does this mean geometrically about the span of these three vectors?

Hint: You might find this computation useful:

> *with*(*LinearAlgebra*) :

$$\text{ReducedRowEchelonForm}\left(\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 6 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right]\right);$$

$$\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

$$\left[\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \qquad \textbf{(1)}$$

$$\underline{c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}}$$

$$-2\vec{v}_1 - 3\vec{v}_2 + \vec{v}_3 = \vec{0}$$

$$\vec{v}_3 = 2\vec{v}_1 + 3\vec{v}_2$$

$c_1 = -2t$

$c_2 = -3t$

$c_3 = t$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \qquad \text{Let } t = 1$$

$\vec{v}_3$ was in the plane, spanned by $\vec{v}_1, \vec{v}_2$

$-2x - y + z = 0$

$2 - 6 + 4 = 0$

Exercise 6) Are the vectors

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ \underline{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent? What is their span? Hint:

> *ReducedRowEchelonForm*$\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}\right)$;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textbf{(2)}$$

Math 2250-004
Fri Feb 24
4.1 - 4.3 Concepts related to linear combinations of vectors.

Friday: mostly doing Wednesday material
I'll post next week's notes by 2:00 p.m.

Exercise 1) Vocabulary review (these need to be memorized!)

A linear combination of the vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is any sum of scalar multiples

i.e. any $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$

The span of $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ is

$\text{span}\{\vec{v}_1\} = \{t\vec{v}_1 \text{ such that } t \in \mathbb{R}\}$
position vectors of a line thru origin

$\text{span}\{\vec{v}_1, \vec{v}_2\} = \{s\vec{v}_1 + t\vec{v}_2, \text{ such that } s, t \in \mathbb{R}\}$
position vectors of a plane thru $\vec{0}$

$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ = all possible linear combo's of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$
$= \{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$ with $c_1, c_2, \ldots c_n \in \mathbb{R}\}$

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly independent iff

The vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ are linearly dependent iff

• Keep recalling that for vectors in $\mathbb{R}^m$ all linear combination questions can be reduced to matrix questions because any linear combination like the one on the left is actually just the matrix product on the right:

$$\underbrace{c_1\begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix} + c_2\begin{bmatrix} a_{12} \\ a_{22} \\ : \\ a_{m2} \end{bmatrix} + \ldots + c_n\begin{bmatrix} a_{1n} \\ a_{2n} \\ 0 \\ a_{mn} \end{bmatrix}}_{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n} = \begin{bmatrix} c_1a_{11} + c_2a_{12} + \ldots \\ c_1a_{21} + c_2a_{22} + \ldots \\ : \\ c_1a_{m1} + c_2a_{m2} + \ldots \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix}$$