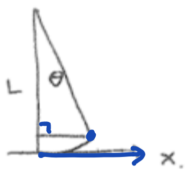
Setup: an ~~underemployed~~ dedicated mathematician/engineer/scientist (your choice)

likes to take his/her child to the swings...

$L\,\theta''(t) + g\sin\theta(t) = 0$

recall pendulum (linearized) eqtn, without forcing, for $\theta = \theta(t)$

$L\theta'' + g\theta = 0$



$\dashrightarrow \; x'' + g\frac{x}{L} = 0$

$\dashrightarrow \; m x'' + \frac{mg}{L} x = F_0 \cos\omega t \quad \leftarrow$ parent forcing (?!)

- $x(t) = L\sin\theta(t) \approx L\theta$ for small θ

so $x'' \approx L\theta''$

$\longrightarrow \quad x'' + \frac{g}{L}x = \frac{F_0}{m}\cos\omega t \qquad \omega = \omega_0$

$x'' + \omega_0^2 x = \underbrace{\frac{F_0}{m}}_{.2}\cos\omega t$

parent pushes sinusoidally for exactly 5 cycles, and with $\frac{F_0}{m} = .2$ and then releases:

for resonance $\omega = \omega_0$ •

construct swing with $L = $ "g" ≈ 9.8 m.

so $\omega_0^2 = 1$, $T_0 = 2\pi \approx 6.2$ seconds

Exercise 3a) Explain why the description above leads to the differential equation initial value problem for $x(t)$

$$x''(t) + x(t) = .2\cos(t)\,(1 - u(t - 10\pi)) \; \bullet$$
$$x(0) = 0$$
$$x'(0) = 0$$

3b) Find $x(t)$. Show that after the parent stops pushing, the child is oscillating with an amplitude of exactly π meters (in our linearized model).

$$x'' + x(t) = \begin{cases} .2\cos t & 0 \le t < 10\pi \\ 0 & t \ge 10\pi \end{cases} \qquad \omega_0 = 1, \; \omega = 1$$

want forcing "on" for $0 \le t < 10\pi$ & then "off" afterwards.

$\text{RHS} = f(t)\big[\underbrace{u(t) - u(t-10\pi)}\big]$

$0 \le t < 10\pi \quad \sim = 1 - 0$

$t \ge 10\pi \quad \sim = 1 - 1 = 0$

• $f(t-a)u(t-a)$	$e^{-as}F(s)$
$x''(t)$	$s^2X(s) - s\,x(0) - x'(0)$
$\cos kt$	$\frac{s}{s^2+k^2}$

IVP $\begin{cases} x'' + x(t) = .2\cos t\,[1 - u(t-10\pi)] \\ x(0) = 0 \\ x'(0) = 0 \end{cases}$

$= .2\cos t - .2\cos t\,(u(t-10\pi))$

✓ $= .2\cos t - .2\cos(t-10\pi)\,u(t-10\pi)$

$\mathcal{L}$: $s^2X(s) + X(s) = .2\frac{s}{s^2+1} - .2e^{-10\pi s}\frac{s}{s^2+1}$

$X(s)(s^2+1) = $

$$\boxed{X(s) = .2\frac{s}{(s^2+1)^2} - .2e^{-10\pi s}\frac{s}{(s^2+1)^2}}$$

$$x(t) = .2\,\frac{t}{2}\sin t - u(t-10\pi)\left[.2\,\frac{(t-10\pi)}{2}\sin t\right]$$

Pictures for the swing:

$f(t) = \frac{t}{2}\sin t$

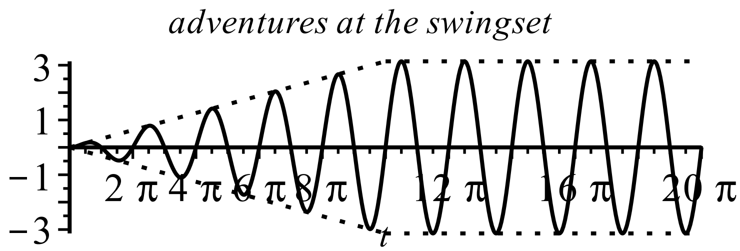
$a = 10\pi$

$f(t-a) = \frac{(t-10\pi)}{2}\underbrace{\sin(t-10\pi)}_{\sin t}$

$f(t)$	$F(s)$
$\frac{t}{2k}\sin kt$	$\frac{s}{(s^2+k^2)^2}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

$$x(t) = \begin{cases} .1\,t\sin t & 0 \le t < 10\pi \\ .1t\cancel{\sin t} - .1(\cancel{t}-10\pi)\sin t = \pi\sin t, & t \ge 10\pi \end{cases}$$

```
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `adventures at the swingset`);
```



$.1t\sin t$ $\qquad$ $\pi\sin t$

Alternate approach via Chapter 5:

<u>step 1)</u> solve

$$x''(t) + x(t) = .2\cos(t)$$
$$x(0) = 0$$
$$x'(0) = 0$$

for $0 \le t \le 10\pi$.

<u>step 2)</u> Then solve

$$y''(t) + y(t) = 0$$
$$y(0) = x(10\pi)$$
$$y'(0) = x'(10\pi)$$

and set $x(t) = y(t-10)$ for $t > 10$.

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	↓ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\frac{1}{s}$ $(s > 0)$	☐
t	$\frac{1}{s^2}$	☐
t^2	$\frac{2}{s^3}$	☐
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	☐
$e^{\alpha t}$	$\frac{1}{s-\alpha}$ $(s > \Re(a))$	☐
$\cos(kt)$	$\frac{s}{s^2+k^2}$ $(s > 0)$	☐
$\sin(kt)$	$\frac{k}{s^2+k^2}$ $(s > 0)$	☐
$\cosh(kt)$	$\frac{s}{s^2-k^2}$ $(s > k)$	☐
		☐
$\sinh(kt)$	$\frac{k}{s^2-k^2}$ $(s > k)$	☐
$e^{at}\cos(kt)$	$\frac{(s-a)}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$	☐
$e^{at}f(t)$	$F(s-a)$	☐
$u(t-a)$	$\frac{e^{-as}}{s}$	
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	
$\delta(t-a)$	e^{-as}	
$f'(t)$	$s\,F(s) - f(0)$	☐
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$	☐
$f^{(n)}(t),\ n \in \mathbb{N}$	$s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐

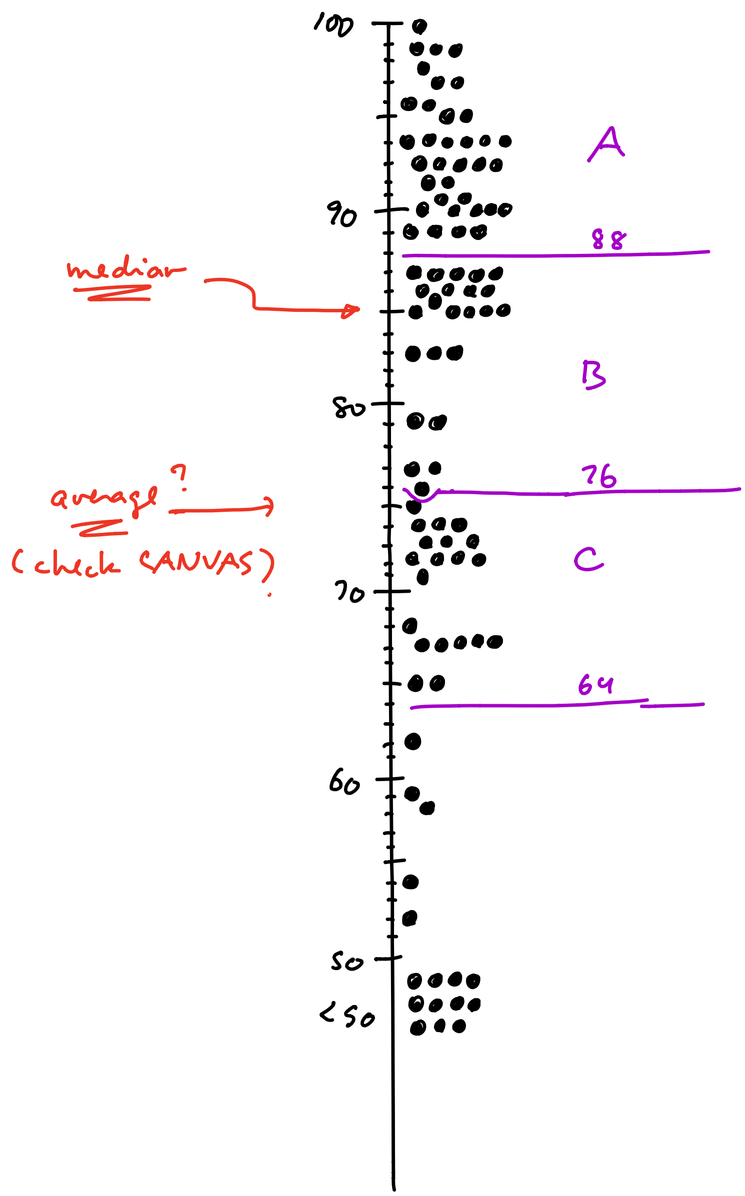
dual

dual

$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	☐
$t\, f(t)$ $t^2\, f(t)$ $t^n\, f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\, F^{(n)}(s)$ $\int_s^\infty F(\sigma)\, d\sigma$	☐ ☐ ☐
$t\cos(k\,t)$ $\frac{1}{2\,k}\, t\sin(k\,t)$ $\frac{1}{2\,k^3}(\sin(k\,t) - k\,t\cos(k\,t))$ $t\, e^{a\,t}$ $t^n\, e^{a\,t},\ n \in \mathbb{Z}$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\frac{1}{1-e^{-ps}}\int_0^p f(t) e^{-s\,t}\, dt$	

Laplace transform table

Math 2250-4
Exam 2 scores
n=94



Fri • on-off example in Wed notes.
• impulse forcing & delta fcn in today's notes

Monday • convolutions

Math 2250-4
Fri Apr 7
10.5, EP7.6 continuing
Today we ~~finish discussing~~ Laplace transform techniques:

- Impulse forcing ("delta functions")...today's notes.
- Convolution formulas to solve any inhomogeneous constant coefficient linear DE, with applications to interesting forced oscillation problems...today's notes.

Laplace table entries for today:

$f(t)$ with $\lvert f(t)\rvert \le C e^{M t}$	$F(s) := \int_0^\infty f(t) e^{-s t}\, dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-a s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\, u(t-a)$	$e^{-a s} F(s)$	more complicated on/off
$\delta(t-a)$	$e^{-a s}$	unit impulse/delta "function"
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	Monday $F(s) G(s)$	convolution integrals to invert Laplace transform products

EP 7.6 impulse functions and the δ operator.

Consider a force $f(t)$ acting on an object for only on a very short time interval $a \le t \le a + \varepsilon$, for example as when a bat hits a ball. This impulse p of the force is defined to be the integral

$$p := \int_a^{a+\varepsilon} f(t)\, dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$m\, v'(t) = f(t)$$

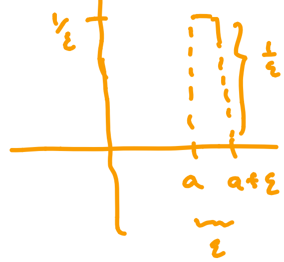
$$\Rightarrow \int_a^{a+\varepsilon} m\, v'(t)\, dt = \int_a^{a+\varepsilon} f(t)\, dt = p$$

$$\Rightarrow m\, v(t)\Big]_{t=a}^{a+\varepsilon} = p\,.$$

Since the impulse p only depends on the integral of $f(t)$, and since the exact form of f is unlikely to be known in any case, the easiest model is to replace f with a constant force having the same total impulse, i.e. to set

$$f = p\, d_{a,\varepsilon}(t)$$

where $d_{a,\varepsilon}(t)$ is the unit impulse function given by



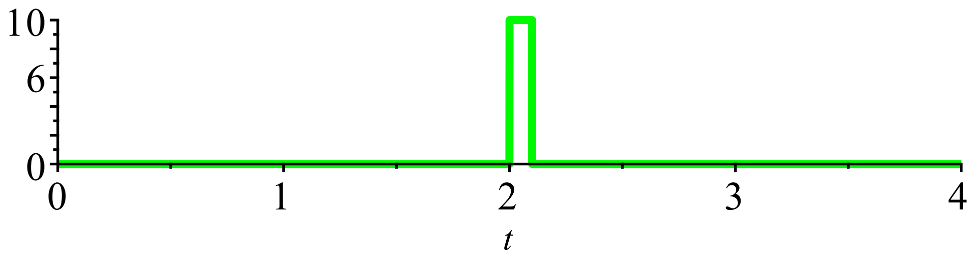
$$d_{a,\varepsilon}(t) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a \le t < a+\varepsilon \\ 0, & t \ge a+\varepsilon \end{cases}.$$

$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u(t-(a+\varepsilon))\right]$

Notice that

$$\int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\, dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, dt = 1 .$$

Here's a graph of $d_{2,\,.1}(t)$, for example:



Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as $\varepsilon \to 0$ for the Laplace transforms $\mathcal{L}\{d_{a,\varepsilon}(t)\}(s)$, and this effectively models impulses on very short time scales.

$$d_{a,\varepsilon}(t) = \frac{1}{\varepsilon}\left[u(t-a) - u(t-(a+\varepsilon))\right]$$

$$\Rightarrow \mathcal{L}\{d_{a,\varepsilon}(t)\}(s) = \frac{1}{\varepsilon}\left(\frac{e^{-a s}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$\lim_{\varepsilon \to 0} \quad = e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right).$$

$f(t-a)u(t-a)$	$e^{-as}F(s)$
$u(t-a)$	$e^{-as}\frac{1}{s}$

In Laplace land we can use L'Hopital's rule (in the variable ε) to take the limit as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} e^{-a s}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon s}\right) = e^{-a s}\lim_{\varepsilon \to 0}\left(\frac{s\, e^{-\varepsilon s}}{s}\right) = e^{-a s}.$$

L'Hôpital

$\frac{d}{d\varepsilon}\left(1 - e^{-\varepsilon s}\right) = -e^{-\varepsilon s}(-s)$

$\frac{d}{d\varepsilon}(\varepsilon s) = s$

The result in time t space is not really a function but we call it the "delta function" $\delta(t-a)$ anyways, and visualize it as a function that is zero everywhere except at $t=a$, and that it is infinite at $t=a$ in such a way that its integral over any open interval containing a equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a linear transformation, not as a function. It can also be thought of as the derivative of the unit step function $u(t-a)$, and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

$\delta(t-a)$ unit impulse function	$e^{-a s}$	for impulse forcing

Exercise 1) Revisit the swing from Wednesday's notes and solve the IVP below for $x(t)$. In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$\begin{cases} x''(t) + x(t) = .2\,\pi\left[\delta(t) + \delta(t-2\,\pi) + \delta(t-4\,\pi) + \delta(t-6\,\pi) + \delta(t-8\,\pi)\right] \\ x(0) = 0 \\ x'(0) = 0\,. \end{cases}$$

$\mathcal{L}$: $\quad s^2 X(s) + X(s) = .2\pi\left[1 + e^{-2\pi s} + e^{-4\pi s} + e^{-6\pi s} + e^{-8\pi s}\right]$

$$X(s) = .2\pi\left[\frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-4\pi s}}{s^2+1} + \dots + \frac{e^{-8\pi s}}{s^2+1}\right]$$

$$x(t) = .2\pi\Big[\sin t + u(t-2\pi)\underbrace{\sin(t-2\pi)}_{=\sin t} + u(t-4\pi)\underbrace{\sin(t-4\pi)}_{=\sin t} + \dots + u(t-8\pi)\sin(t-8\pi)\Big]$$

$$x(t) = .2\pi \sin t\left[1 + u(t-2\pi) + u(t-4\pi) + u(t-6\pi) + u(t-8\pi)\right]$$

$t \geq 8\pi$

$$x(t) = .2\pi \sin t\,(5) = \pi \sin t$$

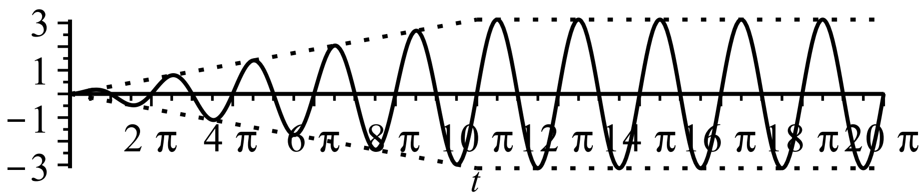
f	F
$\sin kt$	$\frac{k}{s^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$

$F(s) = \frac{1}{s^2+1}$

$f(t) = \sin t$

```
> with(plots) :
> plot1 := plot(.1·t·sin(t), t = 0 ..10·Pi, color = black) :
  plot2 := plot(Pi·sin(t), t = 10·Pi ..20·Pi, color = black) :
  plot3 := plot(Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot4 := plot(−Pi, t = 10·Pi ..20·Pi, color = black, linestyle = 2) :
  plot5 := plot(.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  plot6 := plot(−.1·t, t = 0 ..10·Pi, color = black, linestyle = 2) :
  display({plot1, plot2, plot3, plot4, plot5, plot6}, title = `Wednesday adventures at the swingset`);
```

Wednesday adventures at the swingset

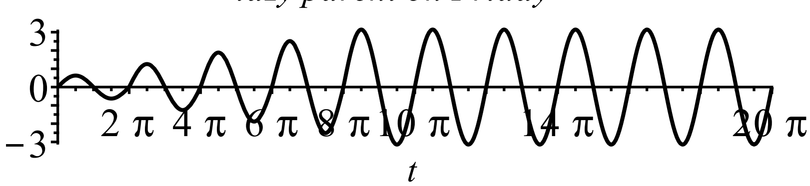



```
>
```

impulse solution: five equal impulses to get same final amplitude of π meters - Exercise 1:

```
> f := t→.2·Pi·sum(Heaviside(t − k·2·Pi)·sin(t − k·2·Pi), k = 0 ..4) :
> plot(f(t), t = 0 ..20·Pi, color = black, title = `lazy parent on Friday`);
```

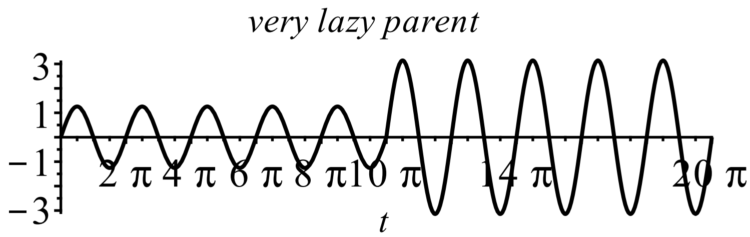
lazy parent on Friday




```
>
```

Or, an impulse at $t=0$ and another one at $t=10\,\pi$.

```
> g := t→.2·Pi·(2·sin(t) + 3·Heaviside(t − 10·Pi)·sin(t − 10·Pi)) :
> plot(g(t), t=0..20·Pi, color=black, title=`very lazy parent`);
```




```
>
```

Convolutions and solutions to non-homogeneous physical oscillation problems (EP7.6 p. 499-501)

Consider a mechanical or electrical forced oscillation problem for $x(t)$, and the particular solution that begins at rest:

$$a\,x'' + b\,x' + c\,x = f(t)$$
$$x(0) = 0$$
$$x'(0) = 0\,.$$

Then in Laplace land, this equation is equivalent to

$$a\,s^2\,X(s) + b\,s\,X(s) + c\,X(s) = F(s)$$
$$\Rightarrow X(s)\left(a\,s^2 + b\,s + c\right) = F(s)$$
$$\Rightarrow X(s) = F(s)\cdot\frac{1}{a\,s^2 + b\,s + c} \;:=\; F(s)W(s)\,.$$

Because of the convolution table entry

$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	convolution integrals to invert Laplace transform products

the solution is given by

$$x(t) = f{*}w(t) = w{*}f(t) = \int_0^t w(\tau)\,f(t-\tau)\,d\tau\,.$$

where $w(t) = \mathcal{L}^{-1}\{W(s)\}(t)$. The function $w(t)$ is called the "weight function" of the differential equation, because the solution $x(t)$ is some sort of weighted average of the the forces f between times 0 and t, where the weighting factors are given by w in some sort of convoluted way.

This idea generalizes to much more complicated mechanical and circuit systems, and is how engineers experiment mathematically with how proposed configurations will respond to various input forcing functions, once they figure out the weight function for their system.

The mathematical justification for the general convolution table entry is at the end of today's notes, for those who have studied iterated double integrals and who wish to understand it.