$\frac{1}{2k}t\sin(kt)$	$\frac{s}{(s^2+k^2)^2}$	$\boxtimes$
$\frac{1}{2k^3}(\sin(kt) - kt\cos(kt))$	$\frac{1}{(s^2+k^2)^2}$	$\boxtimes$
$t\,e^{a t}$	$\frac{1}{(s-a)^2}$	$\boxtimes$
$t^n e^{a t},\ n \in \mathbb{Z}$	$\frac{n!}{(s-a)^{n+1}}$	$\boxtimes$

Laplace transform table

Exercise 6) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4x(t) = 8te^{2t}$$
$$x(0) = 0$$
$$x'(0) = 1$$

$x_p = d_1 t e^{2t} + d_2 e^{2t}$

$\mathcal{L}$:

- $s^2X(s) - s\cdot 0 - 1 + 4X(s) = \frac{8}{(s-2)^2}$
- $X(s)(s^2+4) = \frac{8}{(s-2)^2} + 1$
- $X(s) = \frac{8}{(s-2)^2(s^2+4)} + \frac{1}{s^2+4}$

$f(t)$	$F(s)$
te^{at}	$\frac{1}{(s-a)^2}$
$x'(t)$	$sX(s) - x(0)$
$x''(t)$	$s^2X(s) - s\,x(0) - x'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sin kt$	$\frac{k}{s^2+k^2}$

- $\frac{8}{(s-2)^2(s^2+4)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{Cs+D}{s^2+4}$
- $8 = A(s-2)(s^2+4) + B(s^2+4) + (Cs+D)(s-2)^2$, with (s^2-4s+4)

@ $s=2$: $8 = B\cdot 8 \Rightarrow \boxed{B=1}$

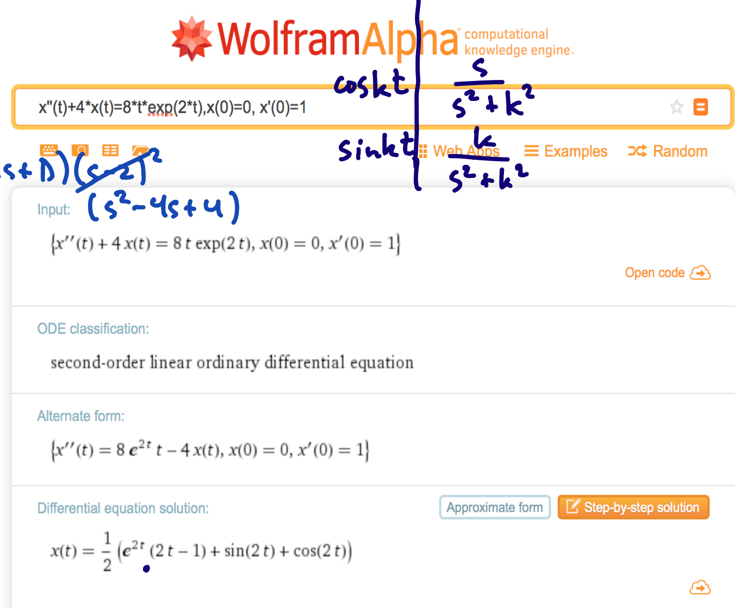
Wed

$$8\cdot 1 + 0s + 0s^2 + 0s^3 = 1(-8A+4+4D) + s(4A+4C-4D) + s^2(-2A+1-4C+D) + s^3(A+C)$$

$8 = -8A+4+4D$
$0 = 4A+4C-4D$
$0 = -2A+1-4C+D$
$0 = A+C$

$2 = -2A+1+\cancel{D} \longrightarrow A = -\frac{1}{2}$
$0 = A+C-\cancel{D}$
$0 = -2A+1-4C+\cancel{D}$
$0 = A+C$

$D=0$
$\rightarrow C = \frac{1}{2}$
$B=1$


WolframAlpha computational knowledge engine.

x"(t)+4*x(t)=8*t*exp(2*t),x(0)=0, x'(0)=1

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Input:
$\{x''(t) + 4x(t) = 8t\exp(2t),\ x(0) = 0,\ x'(0) = 1\}$

Open code

ODE classification:
second-order linear ordinary differential equation

Alternate form:
$\{x''(t) = 8e^{2t}t - 4x(t),\ x(0) = 0,\ x'(0) = 1\}$

Differential equation solution: Approximate form | Step-by-step solution

$x(t) = \frac{1}{2}\left(e^{2t}(2t-1) + \sin(2t) + \cos(2t)\right)$

$$X(s) = -\frac{1}{2}\frac{1}{s-2} + \frac{1}{(s-2)^2} + \frac{1}{2}\frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}$$

$$x(t) = -\frac{1}{2}e^{2t} + te^{2t} + \frac{1}{2}\cos 2t + \frac{1}{2}\sin 2t$$

Wolfram alpha can check most of your steps, once you've set up the problem. Or, if it's a ridiculous problem don't try to even work it by hand:

Exercise 7a) What is the *form* of the partial fractions decomposition for

$$X(s) = \frac{-356 + 45\,s - 100\,s^2 - 4\,s^5 - 9\,s^4 + 39\,s^3 + s^6}{(s-3)^3\left((s+1)^2+4\right)\left(s^2+4\right)^2}.$$

7b) Check exact numbers with Wolfram alpha

7c) What is $x(t) = \mathcal{L}^{-1}\{X(s)\}(t)$?

7d) Have Wolfram alpha compute the inverse Laplace transform directly.

7a) $$X(s) = \frac{A}{s-3} + \frac{B}{(s-3)^2} + \frac{C}{(s-3)^3} + \frac{Ds+E}{(s+1)^2+4} + \frac{Fs+G}{s^2+4} + \frac{Hs+K}{(s^2+4)^2}$$

WolframAlpha computational knowledge engine.

partial fractions

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Assuming "partial fractions" refers to a computation | Use as a general topic instead

rational function: (s^6-4*s^5-9*s^4+39

Input:

partial fractions $\dfrac{s^6-4s^5-9s^4+39s^3-100s^2+45s-356}{(s-3)^3\left((s+1)^2+4\right)(s^2+4)}$

Open code

Result: Step-by-step solution

$$\frac{s^6-4s^5-9s^4+39s^3-100s^2+45s-356}{(s-3)^3(s^2+4)\left((s+1)^2+4\right)} =$$
$$\frac{1}{2(s+2i)} - \frac{i}{2(s+(1-2i))} + \frac{i}{2(s+(1+2i))} + \frac{1}{(s-3)^2} - \frac{4}{(s-3)^3} + \frac{1}{2(s-2i)}$$

$$\frac{1}{2}\left(\frac{1}{s+2i} + \frac{1}{s-2i}\right) = \frac{1}{2}\left(\frac{2s}{s^2+4}\right) = \frac{s}{s^2+4}$$

WolframAlpha computational knowledge engine.

inverse laplace transform

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Assuming "inverse laplace transform" refers to a computation | Use as referring to a mathematical definition instead

function to transform: (s^6-4*s^5-9*s^4+39

initial variable: s

transform variable: t

Input:

$$\mathcal{L}_s^{-1}\left[\frac{s^6-4s^5-9s^4+39s^3-100s^2+45s-356}{(s-3)^3\left((s+1)^2+4\right)(s^2+4)}\right](t)$$

Open code

$\mathcal{L}_s^{-1}[f(s)](t)$ is the inverse Laplace transform of $f(s)$ with real variable t

Result:

$$-2e^{3t}t^2 + e^{3t}t - \frac{1}{2}\,i\,e^{(-1-2i)t}\left(-1+e^{4it}\right) + \cos(2t)$$

Wed
- finish Tues example (partial fracs & IVP's)
- part of today's notes
- quiz: IVP

Math 2250-4
Wed Apr 5

10.4-10.5

• The following Laplace transform material is useful in systems where we turn forcing functions on and off, and when we have right hand side "forcing functions" that are more complicated than what undetermined coefficients can handle. We will continue this discussion on Friday, with a few more table entries including "the delta (impulse) function".

$f(t)$ with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\dfrac{e^{-as}}{s}$	for turning components on and off at $t=a$.
$f(t-a)\,u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\int_0^t f(t-\tau)f(\tau)\,d\tau$	$F(s)G(s)$	"convolution" for inverting products of Laplace transforms

The unit step function with jump at $t=0$ is defined to be

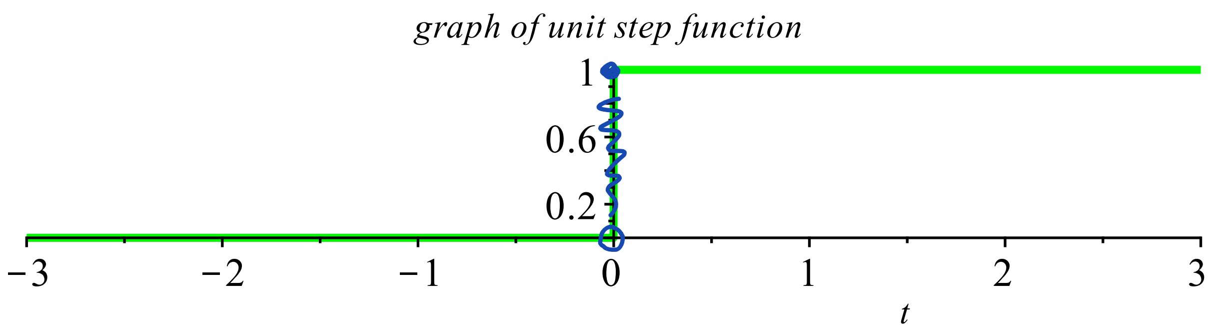
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}.$$

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

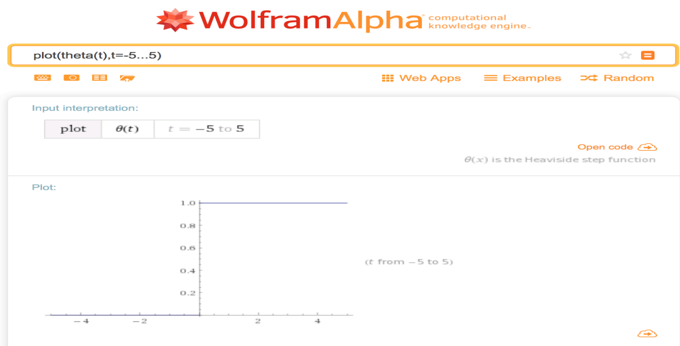
IThis function is also called the "Heaviside" function, e.g. in Maple and Wolfram alpha. In Wolfram alpha it's also called the "theta" function. Oliver Heaviside was a an accomplished physicist in the 1800's. The name is not because the graph is heavy on one side. :-)

http://en.wikipedia.org/wiki/Oliver_Heaviside

```
> with(plots):
  plot(Heaviside(t), t=-3..3, color = green, title = `graph of unit step function`);
```



Notice that technically the vertical line should not be there - a more precise picture would have a solid point at $(0, 1)$ and a hollow circle at $(0, 0)$, for the graph of $u(t)$. In terms of Laplace transform integral definition it doesn't actually matter what we define $u(0)$ to be.

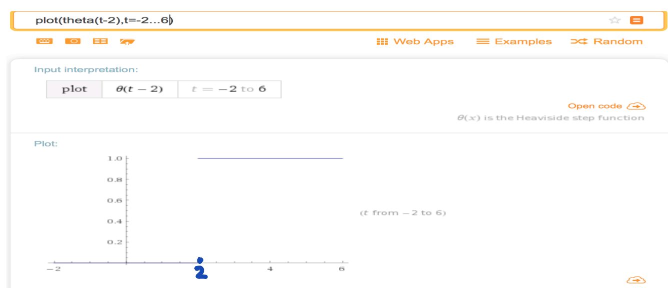


$$u(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

Then

$$u(t-a) = \begin{cases} 0, & t-a < 0;\ i.e.\ t < a \\ 1, & t-a \geq 0;\ i.e.\ t \geq a \end{cases}$$

and has graph that is a horizontal translation by a to the right, of the original graph, e.g. for $a = 2$:



Exercise 1) Verify the table entries

1 · $u(t-a)$ unit step function	$\frac{e^{-a\,s}}{s}$	for turning components on and off at $t = a$.
$f(t-a)\ u(t-a)$	$e^{-a\,s}F(s)$	more complicated on/off

$f = 1$

$F = \frac{1}{s}$

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty f(t-a)u(t-a)\,e^{-st}\,dt = \int_0^a 0\,dt + \int_a^\infty f(t-a)\cdot 1\, e^{-st}\,dt$$

$1 \mid \frac{1}{s}$

$\tilde{t} = t - a \qquad t = \tilde{t} + a$

$d\tilde{t} = dt$

$$= 0 + \int_0^\infty f(\tilde{t})\,e^{-s(\tilde{t}+a)}\,d\tilde{t}$$

$$= \int_0^\infty f(\tilde{t})\,e^{-s\tilde{t}}\,e^{-sa}\,d\tilde{t}$$

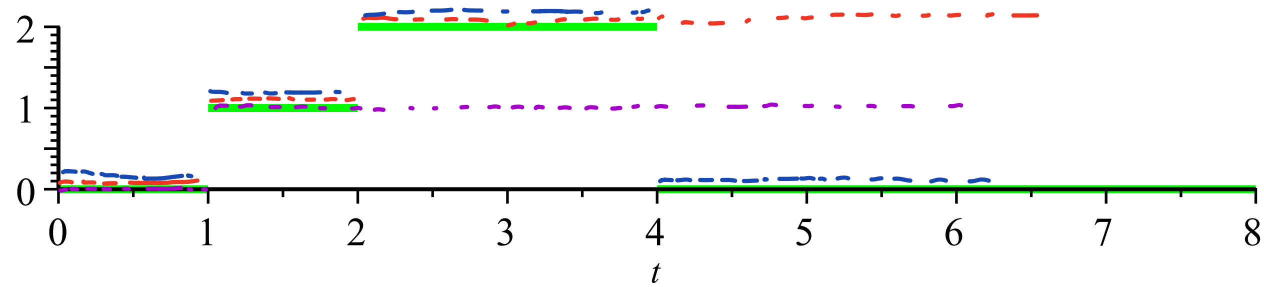
$t = a$

$\tilde{t} = 0$

$$= e^{-as}\int_0^\infty f(t)\,e^{-st}\,dt$$

$$= e^{-as}F(s)$$

Exercise 2) Consider the function $f(t)$ which is zero for $t > 4$ and with the following graph. Use linearity and the unit step function entry to compute the Laplace transform $F(s)$. This should remind you of a homework problem from the assignment due tomorrow - although you're asked to find the Laplace transform of that step function directly from the definition. In your next week's homework assignment you will re-do that problem using unit step functions. (Of course, you could also check your answer in this week's homework with this method.)



takehome quiz

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 2, & 2 \le t < 4 \\ 0, & 4 \le t \end{cases}$$

$$f(t) = u(t-1) + u(t-2) - 2u(t-4)$$

$$\mathcal{L}\{f(t)\}(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - 2\frac{e^{-4s}}{s}$$

$u(t-a)$	e^{-as}/s

$f(t)$, with $\lvert f(t)\rvert \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt$ for $s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n,\ n\in\mathbb{N}$	$\frac{1}{s}\quad (s>0)$ $\frac{1}{s^2}$ $\frac{2}{s^3}$ $\frac{n!}{s^{n+1}}$	☐ ☐ ☐ ☐
$e^{\alpha t}$	$\frac{1}{s-\alpha}\quad (s > \Re(a))$	☐
$\cos(kt)$ $\sin(kt)$ $\cosh(kt)$ $\sinh(kt)$ $e^{at}\cos(kt)$ $e^{at}\sin(kt)$ $e^{at}f(t)$	$\frac{s}{s^2+k^2}\quad (s>0)$ $\frac{k}{s^2+k^2}\quad (s>0)$ $\frac{s}{s^2-k^2}\quad (s>k)$ $\frac{k}{s^2-k^2}\quad (s>k)$ $\frac{(s-a)}{(s-a)^2+k^2}\quad (s>a)$ $\frac{k}{(s-a)^2+k^2}\quad (s>a)$ $F(s-a)$	☐ ☐ ☐ ☐ ☐ ☐ ☐ ☐
$u(t-a)$ $f(t-a)\,u(t-a)$ $\delta(t-a)$	$\frac{e^{-as}}{s}$ $e^{-as}F(s)$ e^{-as}	
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n\in\mathbb{N}$	$s\,F(s) - f(0)$ $s^2F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	☐ ☐ ☐

dual

dual

$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$	☐
$t\, f(t)$ $t^2\, f(t)$ $t^n\, f(t),\ n \in \mathbb{Z}$ $\frac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\, F^{(n)}(s)$ $\int_s^\infty F(\sigma)\, d\sigma$	☐ ☐ ☐
$t\cos(k\,t)$ $\frac{1}{2\,k}\, t\sin(k\,t)$ $\frac{1}{2\,k^3}\,(\sin(k\,t) - k\,t\cos(k\,t))$ $t\, e^{a\,t}$ $t^n\, e^{a\,t},\ n \in \mathbb{Z}$	$\frac{s^2 - k^2}{(s^2 + k^2)^2}$ $\frac{s}{(s^2 + k^2)^2}$ $\frac{1}{(s^2 + k^2)^2}$ $\frac{1}{(s-a)^2}$ $\frac{n!}{(s-a)^{n+1}}$	☐ ☐ ☐ ☐ ☐
$\int_0^t f(\tau) g(t-\tau)\, d\tau$	$F(s)G(s)$	
$f(t)$ with period p	$\frac{1}{1 - e^{-ps}} \int_0^p f(t) e^{-s\,t}\, dt$	

Laplace transform table