$$m_1 x_1'' = -k_1 x_1 + k_2(x_2 - x_1) \quad = \quad -(k_1 + k_2)x_1 + k_2 x_2$$
$$m_2 x_2'' = -k_2(x_2 - x_1) - k_3 x_2 \quad = \quad \frac{k_2 x_1}{m_2} \quad - (k_2 + k_3) x_2$$

We can write the system of DEs for the system at the top of page 1 in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M \underline{x}''(t) = K \underline{x}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\underline{x}''(t) = A \underline{x}.$$

(You can think of A as the "acceleration" matrix.)
Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix M. In all cases:

$$M \underline{x}''(t) = K \underline{x} \quad \Rightarrow \underline{x}''(t) = A \underline{x}, \text{ with } A = M^{-1} K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\mathbf{x}''(t) = A\,\mathbf{x}\,.$$ (*) *Review Monday*

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

substitute

$$\mathbf{x}(t) = f(t)\mathbf{v}$$ (**)

RHS

You might guess that $f(t) = e^{\lambda t}$ but that turns out to not be the best way to go. Let's see what $f(t)$ should equal by substituting in our guess! (We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function, and then the exponential guess would work, but they'd end up being complex exponentials.) Substituting (**) into (*) yields

$$\mathbf{x}''(t) = f''(t)\,\mathbf{v} = A\,(f(t)\mathbf{v}) = f(t)\,A\,\mathbf{v}\,.$$ •

Since for each t, the left side is a scalar multiple of the constant vector $\mathbf{v}$, so must be the right side. So $\mathbf{v}$ must be an eigenvector of A,

$$A\,\mathbf{v} = \lambda\,\mathbf{v}\,,$$

and if $f(t)$ is a real function and if $\mathbf{v}$ is a real (as opposed to complex) vector, then λ is also real. Then

$$f''(t)\,\mathbf{v} = A\,(f(t)\mathbf{v}) = f(t)\,\lambda\,\mathbf{v}$$ • $f''(t) = f(t)\lambda$

So we must have

$$f''(t) - \lambda f(t) = 0.$$

So possible $f(t)$'s are *(depending on λ)*

Case 1)

$$f''(t) = 0 \implies f(t) = c_1 + c_2 t \qquad \text{if } \lambda = 0 \qquad f(t) = (c_1 + c_2 t)$$

Case 2)

$$f(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) \qquad \boxed{\text{if } \lambda < 0,\; \lambda = -\omega^2 \quad \omega = \sqrt{-\lambda}} \qquad f''(t) + (-\lambda) f(t) = 0$$

> 0

Case 3)

$$f(t) = c_1 e^{\sqrt{\lambda}\,t} + c_2 e^{-\sqrt{\lambda}\,t} \qquad\qquad \text{if } \lambda > 0\,. \qquad \boxed{f''(t) + \omega^2 f(t) = 0}$$

$$f''(t) - \lambda f(t) = 0$$

Case 3 will never happen for our mass-spring configurations, because of conservation of energy!

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,
$$\underline{x}''(t) = A\,\underline{x}\ .$$
If $A_{n \times n}$ is a diagonalizable matrix and if all of its eigenvalues are non-positive then for each eigenpair $\left(\lambda_j, \underline{v}_j\right)$ with $\lambda_j < 0$ there are two linearly independent sinusoidal solutions to $\underline{x}''(t) = A\,\underline{x}$ given by
$$\underline{x}_j(t) = \cos\left(\omega_j\,t\right)\underline{v}_j \qquad \underline{y}_j(t) = \sin\left(\omega_j\,t\right)\underline{v}_j$$
with
$$\omega_j = \sqrt{-\lambda_j}\ .$$
And for an eigenpair $\left(\lambda_j, \underline{v}_j\right)$ with $\lambda_j = 0$ there are two indpendent solutions given by constant and linear functions
$$\underline{x}_j(t) = \underline{v}_j \qquad \underline{y}_j(t) = t\,\underline{v}_j$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{x}''(t) = A\,\underline{x}$, i.e. a basis for the solution space.

Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $\left(c_1 + c_2\,t\right)\underline{v}$, where $\underline{v}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)
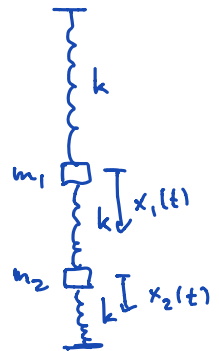
<u>Exercise 2)</u> Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$ In this case, the equation for the vector of the two mass accelerations reduces to

$$\vec{x}\,'' = A\vec{x}$$

if

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{2k}{m} & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \dfrac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

<u>a)</u> Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

<u>b)</u> Deduce the eigendata for the acceleration matrix A which is $\dfrac{k}{m}$ times this matrix.

<u>c)</u> Find the 4- dimensional solution space to this two-mass, three-spring system.

a). $\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = (\lambda+3)(\lambda+1)$

$$\lambda = -3, -1.$$

$E_{\lambda=-1}$ $\begin{bmatrix} -1 & 1 & | & 0 \\ 1 & -1 & | & 0 \end{bmatrix}$ $E_{\lambda=-3}$ $\begin{bmatrix} 1 & 1 & | & 0 \\ 1 & 1 & | & 0 \end{bmatrix}$

$E_{\lambda=-1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $E_{\lambda=-3} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

b)   If $A\vec{v} = \lambda\vec{v}$

what is $(sA)\vec{v} = s(A\vec{v}) = (s\lambda)\vec{v}$

if mult matrix by constant, eigenvectors stays same.
                                        eigenvalues get mult by const

for $\dfrac{k}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$\lambda = -3\dfrac{k}{m}, -1\cdot\dfrac{k}{m}$

$\omega = \sqrt{\dfrac{3k}{m}}, \sqrt{\dfrac{k}{m}}$

$\underset{\omega_2}{}\Vert \qquad \underset{\omega_1}{}\Vert$

c) sol'tns:

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left[c_1 \cos\sqrt{\dfrac{k}{m}}\,t + c_2 \sin\sqrt{\dfrac{k}{m}}\,t \right]\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left[c_3 \cos\left(\sqrt{\dfrac{3k}{m}}\,t\right) + c_4 \sin\left(\sqrt{\dfrac{3k}{m}}\,t\right)\right]\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

"slow"
in-phase

amplitude
-phase for

"fast"
out of phase

solution The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\dfrac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\dfrac{3\,k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos(\omega_1 t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\omega_2 t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t) \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t) \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Exercise 3) Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is four dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{2k}{m} & \dfrac{k}{m} \\ \dfrac{k}{m} & -\dfrac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

units for k $\quad$ N/m $\quad$ kg/s²

$\quad m \quad$ kg

$$\frac{k}{m} = \frac{1}{s^2}$$

Experiment: Although we probably won't have time in class to measure the spring constants, I've measured them earlier. We can predict the numerical values for the two fundamental modes of the vertical mass-spring configuration corresponding to Exercise 2, and then check our predictions like we did for the single mass-spring configuration, I have brought along a demonstration so that we can see these two vibrations.

```
> Digits := 5 :
```

$$k := \frac{.05 \cdot 9.806}{.153};$$

$$\omega 1 := \sqrt{\frac{k}{.05}} \; ; \; T1 := evalf\left(\frac{2 \cdot \pi}{\omega 1}\right);$$

$$\omega 2 := \sqrt{3.0} \cdot \omega 1; \; T2 := evalf\left(\frac{2 \cdot \pi}{\omega 2}\right);$$

$$k := 3.2046$$

$$\omega 1 := 8.0057$$

$$T1 := 0.78483$$

$$\omega 2 := 13.867$$

$$T2 := 0.45311 \tag{2}$$

<u>Exercise 4)</u> Consider a train with two cars connected by a spring:
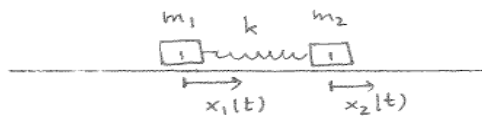
$$m_1 x_1'' = \frac{k}{m_1}(x_2 - x_1)$$

$$m_2 x_2'' = -\frac{k}{m_2}(x_2 - x_1)$$

<u>4a)</u> Verify that the linear system of DEs that governs the dynamics of this configuration (it's actually a special case of what we did before, with two of the spring constants equal to zero) is

$$x_1'' = \frac{k}{m_1}(x_2 - x_1)$$

$$x_2'' = -\frac{k}{m_2}(x_2 - x_1)$$

✓

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

<u>4b)</u>. Use the eigenvalues and eigenvectors computed below to find the general solution. For $\lambda = 0$ and its corresponding eigenvector $\underline{v}$ remember that you get two solutions

$$\underline{x}(t) = \underline{v} \text{ and } \underline{x}(t) = t\,\underline{v},$$

rather than the expected $\cos(\omega t)\underline{v}$, $\sin(\omega t)\underline{v}$. Interpret these solutions in terms of train motions. You will use these ideas in some of your homework problems and in your lab exercise about molecular vibrations.

$$= \begin{bmatrix} -c & c \\ d & -d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

> *Eigenvectors* $\left(\left[\begin{bmatrix} -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{k}{m_2} & -\frac{k}{m_2} \end{bmatrix}\right]\right)$;

$$\boxed{\lambda = 0,} \quad E_{\lambda=6} = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$$

$$\boxed{\lambda = \frac{-k(m_1 + m_2)}{m_1 m_2}} \quad E_{\lambda = *} = \text{span}\left\{\begin{bmatrix} -m_2 \\ m_1 \end{bmatrix}\right\}.$$

$$\begin{bmatrix} 0 \\ -\frac{k(m_1 + m_2)}{m_2 m_1} \end{bmatrix}, \begin{bmatrix} 1 & -\frac{m_2}{m_1} \\ 1 & 1 \end{bmatrix}$$

$$\lambda = 0$$

(3)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 + c_2 t)\begin{bmatrix}1\\1\end{bmatrix} + (c_3 \cos\omega_2 t + c_4 \sin\omega_2 t)\begin{bmatrix} -m_2 \\ m_1 \end{bmatrix}$$

train started
@ c_1, moves
with velocity c_2
(no oscillation)

$$\left(\omega_2 = \sqrt{-\lambda_2}\right)$$

out of phase

Math 2250-004
Week 15: Section 7.4, mass-spring systems.
These are notes for Monday (and possibly part of Tuesday). There will also be course review notes for Tuesday, posted later.

Mon Apr 24
7.4 Mass-spring systems and untethered mass-spring trains, continued. First do the warmup exercise, then finish Friday's notes about unforced oscillations and natural frequencies/modes. Today's notes are about forced oscillation problems.

Warm up exercise: Here are two systems of differential equations, and the eigendata is as shown. The first order system could arise from an input-output model, and the second one could arise from an undamped two mass, three spring model. Write down the general solution to each system.

1a)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1b)

$$\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

eigendata: For the matrix

$$\begin{bmatrix} -3 & 4 \\ 1 & -3 \end{bmatrix}$$

$\omega = \sqrt{5}$

$\omega = 1$

for the eigenvalue $\lambda = -5$, $\underline{v} = [-2, 1]^T$ is an eigenvector; for the eigenvalue $\lambda = -1$, $\underline{v} = [2, 1]^T$ is an eigenvector

1a)
(e.g. input-output model)
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-5t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

dim of soln space = 2
for the IVP specify $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

1b)
(e.g. undamped mass-spring configuration)
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = (c_1 \cos\sqrt{5}t + c_2 \sin\sqrt{5}t) \begin{bmatrix} -2 \\ 1 \end{bmatrix} + (c_3 \cos t + c_4 \sin t) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$C_1 \cos(\sqrt{5}t - \alpha_1)$

faster "out of phase" oscillation
m_1 has twice amplitude of m_2

slower "in phase oscillation, m_1 has twice amplitude of m_2

dim soln space = 4
for IVP specify
$x_1(0)$
$x_1'(0)$
$x_2(0)$
$x_2'(0)$

Forced oscillations (still undamped):

$$M\underline{x}''(t) = K\underline{x} + \underline{F}(t)$$
$$\Rightarrow \underline{x}''(t) = A\underline{x} + M^{-1}\underline{F}(t) . \leftarrow$$

If the forcing is sinusoidal,

$$M\underline{x}''(t) = K\underline{x} + \boxed{\cos(\omega t)\underline{G}_0}$$
$$\Rightarrow \underline{x}''(t) = A\underline{x} + \cos(\omega t)\underline{F}_0$$

with $\underline{F}_0 = M^{-1}\underline{G}_0$.

$$\vec{x}' - A\vec{x} = \cos\omega t\ \vec{F}_0$$
$$\uparrow$$
non-homog.

From the fundamental theorem for linear transformations we know that the general solution to this inhomogeneous linear problem is of the form

$$\boxed{\underline{x}(t) = \underline{x}_P(t) + \underline{x}_H(t) ,}$$

and we've been discussing how to find the homogeneous solutions $\underline{x}_H(t)$.

As long as the driving frequency ω is NOT one of the natural frequencies, we don't expect resonance; the method of undetermined coefficients predicts there should be a particular solution of the form

$$\underline{x}_P(t) = \cos(\omega t)\ \vec{c}$$

where the vector $\underline{c}$ is what we need to find.

Exercise 1) Substitute the guess $\underline{x}_P(t) = \cos(\omega t)\ \vec{c}$ into the DE system

$$\boxed{\underline{x}''(t) = A\underline{x} + \cos(\omega t)\underline{F}_0}$$

to find a matrix algebra formula for $\underline{d} = \underline{d}(\omega)$. Notice that this formula makes sense precisely when ω is NOT one of the natural frequencies of the system.

$$\vec{x}_p = \cos\omega t\ \vec{d}$$

want $\quad$ LHS $\quad \vec{x}_p'' = -\omega^2 \cos\omega t\ \vec{d}$ want
$\quad\quad\quad\quad$ ‖
$\quad\quad$ RHS $\quad A(\cos\omega t\ \vec{d}) + \cos\omega t\ \vec{F}_0$

need : $\quad -\omega^2 \vec{d} = A\vec{d} + \vec{F}_0$

$\quad\quad\quad -\vec{F}_0 = A\vec{d} + \omega^2 \vec{d} = (A + \omega^2 I)\vec{d}$

$\quad\quad\quad\quad -(A+\omega^2 I)^{-1}\vec{F}_0 = \vec{d} \quad$ our $\vec{d}$

our $\vec{x}_p(t) = \cos\omega t\ \vec{d}$

$$\vec{x}(t) = \vec{x}_p + \vec{x}_H$$

Solution:

$$\underline{c}(\omega) = -(A + \omega^2 I)^{-1}\underline{F}_0 .$$

Note, matrix inverse exists precisely if $-\omega^2$ is not an eigenvalue.

<u>Exercise 2)</u> Continuing with the configuration from page 1 of Monday's notes, but now for an inhomogeneous forced problem, let $k = m$, and force the second mass sinusoidally:

$$\begin{bmatrix} x_1{}''(t) \\ x_2{}''(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \cos(\omega t) \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

We know from previous work that the natural frequencies are $\omega_1 = 1$, $\omega_2 = \sqrt{3}$ and that

$$\underline{x}_H(t) = C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\,t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find the formula for $\underline{x}_P(t)$, as on the preceding page. Notice that this steady periodic solution blows up as $\omega \to 1$ or $\omega \to \sqrt{3}$. (If we don't have time to work this by hand, we may skip directly to the technology check on the next page. But since we have quick formulas for inverses of 2 by 2 matrices, this is definitely a computation we could do by hand.)

<u>Solution</u>: As long as $\omega \neq 1, \sqrt{3}$, the general solution $\underline{x} = \underline{x}_P + \underline{x}_H$ is given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \cos(\omega t) \begin{bmatrix} \dfrac{3}{(\omega^2 - 1)(\omega^2 - 3)} \\ \dfrac{6 - 3\omega^2}{(\omega^2 - 1)(\omega^2 - 3)} \end{bmatrix}$$

$$+ \; C_1 \cos(t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\sqrt{3}\,t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

<u>Interpretation</u> as far as inferred <u>practical resonance</u> for <u>slightly damped</u> problems: If there was even a small amount of damping, the homogeneous solution would actually be transient (it would be exponentially decaying and oscillating - underdamped). There would still be a sinusoidal particular solution, which would have a formula close to our particular solution, the first term above, as long as $\omega \neq 1, \sqrt{3}$. (There would also be a relatively smaller $\sin(\omega t)\underline{d}$ term as well.) So we can infer the practical resonance behavior for different ω values with slight damping, by looking at the size of the $\underline{c}(\omega)$ term for the undamped problem....see next page for visualizations.

```
> restart :
> with(LinearAlgebra) :
> A := Matrix(2, 2, [−2, 1, 1, −2]) :
> F0 := Vector([0, 3]) :
> Iden := IdentityMatrix(2) :
> c := ω→(A + ω²·Iden)⁻¹.(−F0) :   # the formula we worked out by hand
> c(ω);
```

$$\begin{bmatrix} \dfrac{3}{\omega^4 - 4\,\omega^2 + 3} \\[4mm] -\dfrac{3\,(\omega^2 - 2)}{\omega^4 - 4\,\omega^2 + 3} \end{bmatrix}$$
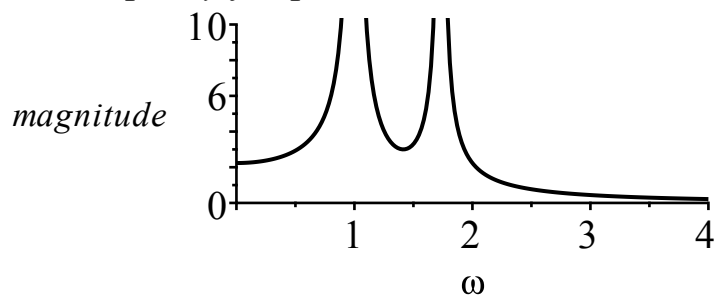
(1)

```
> with(plots) :
  with(LinearAlgebra) :
> plot(Norm(c(ω), 2), ω = 0 ..4, magnitude = 0 ..10, color = black, title
     = `undamped particular solution amplitudes as proxy for practical resonance`);
     # Norm(c(ω),2) is the magnitude of the c(ω) vector
```
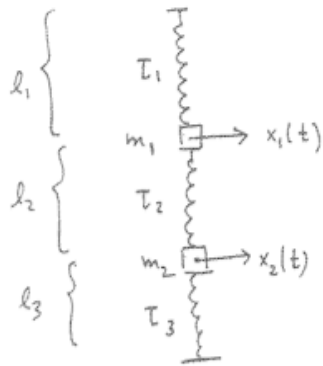

undamped particular solution amplitudes
as proxy for practical resonance

```
>
```

There are strong connections between our discussion here and the modeling of how earthquakes can shake buildings:

- Transverse oscillations! (i.e. directions $\perp$ to the mass-spring configuration)

T_1, T_2, T_3 are the tensions (forces) $\overset{\text{pulling}}{}$ of the stretched springs

By linearization, a good model would be

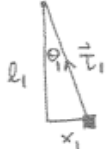$$m_1 x_1'' = -K_1 x_1 + K_2(x_2 - x_1) = -(K_1 + K_2)x_1 + K_2 x_2$$
$$m_2 x_2'' = K_2(x_1 - x_2) - K_3 x_2 = K_2 x_1 - (K_2 + K_3)x_2$$

where K_1, K_2, K_3 are positive constants as before

$\rightarrow$ but in general **not** the Hooke's constants, because to first order the springs are not being stretched beyond their equilibrium lengths in this model.

- upshot: transverse oscillations satisfy analogous systems of 2nd order linear DE's; forcing and resonance will also be analogous to longitudinal vibrations, but probably with different resonant frequencies & fundamental modes.

As it turns out, for our physics lab springs, the modes and frequencies are almost identical:

[force picture, e.g.

horiz force from top spring on mass 1

$$= -T_1 \sin\theta_1 = -T_1 \frac{x_1}{\sqrt{\ell_1^2 + x_1^2}} \approx -T_1 \frac{x_1}{\ell_1} = -\frac{T_1}{\ell_1} x_1$$

$$\text{So } K_1 = \frac{T_1}{\ell_1}$$

similarly, $K_2 = \frac{T_2}{\ell_2}, K_3 = \frac{T_3}{\ell_3}$

for our physics demo springs, equilibrium length ≈ 0, very Hookesian so $T \cong k\ell$; $\frac{T}{\ell} \approx k$, so actually almost recover same fundamental modes !!]

- An interesting shake-table demonstration:

Below is a discussion of how to model the unforced "three-story" building shown shaking in the video above, from which we can see which modes will be excited. There is also a "two-story" building model in the video, and its matrix and eigendata follow. Here's a schematic of the three-story building:
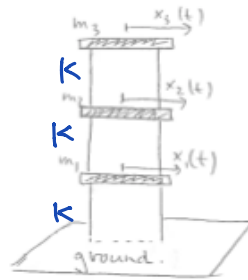
$$m_3 x_3'' = -k(x_3 - x_2) = \frac{k}{m}(x_2 - x_3)$$

$$m_2 x_2'' = -k(x_2 - x_1) + k(x_3 - x_2) = \frac{k}{m}(x_1 - 2x_2 + x_3)$$

$$m x_1'' = -kx_1 + k(x_2 - x_1)$$
$$= \frac{k}{m}(-2x_1 + x_2)$$

For the unforced (homogeneous) problem, the accelerations of the three massive floors (the top one is the roof) above ground and of mass m, are given by

$$
\begin{bmatrix} x_1''(t) \\ x_2''(t) \\ x_3''(t) \end{bmatrix}
= \frac{k}{m}
\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.
$$

← 1st floor
← 2nd floor
← 3rd floor

Note the -1 value in the last diagonal entry of the matrix. This is because $x_3(t)$ is measuring displacements for the top floor (roof), which has nothing above it. The "k" is just the linearization proportionality factor, and depends on the tension in the walls, and the height between floors, etc, as discussed on the previous page.

Here is eigendata for the <u>unscaled matrix</u> $\left(\frac{k}{m} = 1 \right)$. For the scaled matrix you'd have the same eigenvectors, but the eigenvalues would all be multiplied by the scaling factor $\frac{k}{m}$ and the natural frequencies would all be scaled by $\sqrt{\frac{k}{m}}$ but the eigenvectors describing the modes would stay the same.

```
eigenvalues{{-2.0,1,0},{1,-2,1},{0,1,-1}};
```

Input:

$$\text{eigenvalues} \begin{pmatrix} -2. & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Open code

Results: Step-by-step solution

$\lambda_1 \approx -3.24698$

$\lambda_2 \approx -1.55496$

$\lambda_3 \approx -0.198062$

Corresponding eigenvectors: Step-by-step solution

$v_1 \approx (0.591009, -0.736976, 0.327985)$

$v_2 \approx (-0.736976, -0.327985, 0.591009)$

$v_3 \approx (0.327985, 0.591009, 0.736976)$


```
eigenvalues{{-2.0,1},{1,-1}};
```

Input:

$$\text{eigenvalues} \begin{pmatrix} -2. & 1 \\ 1 & -1 \end{pmatrix}$$

Open code

Results: Step-by-step solution

$\lambda_1 \approx -2.61803$

$\lambda_2 \approx -0.381966$

Corresponding eigenvectors: Step-by-step solution

$v_1 \approx (-0.850651, 0.525731)$

$v_2 \approx (-0.525731, -0.850651)$

Exercise 3) Interpret the data above, in terms of the natural modes for the 3-story and 2-story shaking buildings. In the youtube video the first mode to appear is the slow and dangerous "sloshing mode", where all three floors oscillate in phase, with amplitude ratios .33 : .59 : .74 from the first to the third floor. What's the second mode that gets excited? The third mode? (They don't show the third mode in the video.)

Remark) All of the ideas we've discussed in section 7.4 also apply to molecular vibrations. The eigendata in these cases is related to the "spectrum" of light frequencies that correspond to the natural fundamental modes for molecular vibrations. See your lab question about the fundamental modes of carbon dioxide.