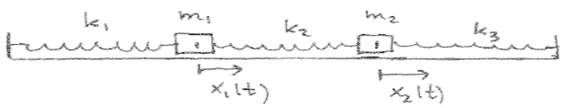
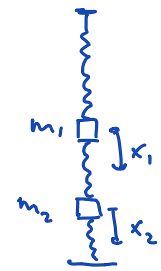
Fri Apr 21

7.4 Mass-spring systems and untethered mass-spring trains.

In your homework and lab for this week you study special cases of the spring systems below, with no damping. Although we draw the pictures horizontally, they would also hold in vertical configuration if we measure displacements from equilibrium in the underlying gravitational field.





Let's make sure we understand why the natural system of DEs and IVP for this system is

0)
$$\begin{cases} m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2 \\ x_1(0) = a_1, \quad x_1'(0) = a_2 \\ x_2(0) = b_1, \quad x_2'(0) = b_2 \end{cases}$$

Exercise 1a) What is the dimension of the solution space to this homogeneous linear system of differential equations? Why?

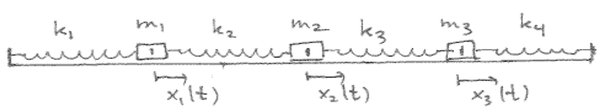
0) $m_1 x_1''(t) = F_{spring\,1} + F_{spring\,2} = -k_1 x_1 + k_2(x_2 - x_1)$

$m_2 x_2'' = F_{spring\,2} + F_{spring\,3} = -k_2(x_2 - x_1) - k_3 x_2$

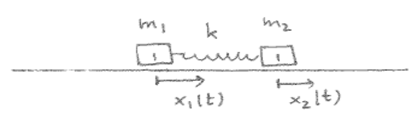
dim = 4 (4 free parameters)

(equiv to system of 4 1st order homog. DE's)

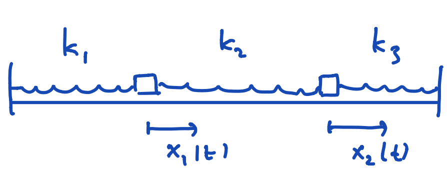
1b) What if one had a configuration of n masses in series, rather than just 2 masses? What would the dimension of the homogeneous solution space be in this case? Why? Examples:



dim = 6



dim = 4



$$\cancel{m_1} x_1'' = -k_1 x_1 + k_2(x_2 - x_1) = -\frac{(k_1+k_2)}{m_1} x_1 + \frac{k_2}{m_1} x_2$$

$$\cancel{m_2} x_2'' = -k_2(x_2 - x_1) - k_3 x_2 = \frac{k_2 x_1}{m_2} - \frac{(k_2+k_3)}{m_2} x_2$$

We can write the system of DEs for the system at the top of page 1 in matrix-vector form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 & k_2 \\ k_2 & -k_2 - k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We denote the diagonal matrix on the left as the "mass matrix" M, and the matrix on the right as the spring constant matrix K (although to be completely in sync with Chapter 5 it would be better to call the spring matrix $-K$). All of these configurations of masses in series with springs can be written as

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}}.$$

If we divide each equation by the reciprocal of the corresponding mass, we can solve for the vector of accelerations:

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{k_1 + k_2}{m_1} & \dfrac{k_2}{m_1} \\ \dfrac{k_2}{m_2} & -\dfrac{k_2 + k_3}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

which we write as

$$\boxed{\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}.}$$

(You can think of A as the "acceleration" matrix.)

Notice that the simplification above is mathematically identical to the algebraic operation of multiplying the first matrix equation by the (diagonal) inverse of the diagonal mass matrix M. In all cases:

$$M\,\underline{\mathbf{x}}''(t) = K\,\underline{\mathbf{x}} \quad \Rightarrow \underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}\ , \text{ with } A = M^{-1}K.$$

How to find a basis for the solution space to conserved-energy mass-spring systems of DEs

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}\ . \qquad (*)$$

Based on our previous experiences, the natural thing for this homogeneous system of linear differential equations is to try and find a basis of solutions of the form

substitute

$$\boxed{\underline{\mathbf{x}}(t) = f(t)\,\underline{\mathbf{v}}} \qquad (**)$$

LHS RHS of *

You might guess that $f(t) = e^{\lambda t}$ but that turns out to not be the best way to go. Let's see what $f(t)$ should equal by substituting in our guess! (We would maybe also think about first converting the second order system to an equivalent first order system of twice as many DE's, one for for each position function and one for each velocity function, and then the exponential guess would work, but they'd end up being complex exponentials.) Substituting (**) into (*) yields

$$\vec{x}''(t) = \boxed{f''(t)\,\underline{\mathbf{v}} = A\,(f(t)\underline{v}) = f(t)\,A\,\underline{\mathbf{v}}\,.}$$

Since for each t, the left side is a scalar multiple of the constant vector $\underline{\mathbf{v}}$, so must be the right side. So $\underline{\mathbf{v}}$ must be an eigenvector of A,

$$A\,\underline{\mathbf{v}} = \lambda\,\underline{\mathbf{v}}\,,$$

and if $f(t)$ is a real function and if $\underline{\mathbf{v}}$ is a real (as opposed to complex) vector, then λ is also real. Then

$$\boxed{f''(t)\,\underline{\mathbf{v}} = A\,(f(t)\underline{v}) = f(t)\,\lambda\,\underline{\mathbf{v}}} \qquad f''(t) = f(t)\lambda$$

So we must have

$$f''(t) - \lambda f(t) = 0.$$

So possible $f(t)$'s are (depending on λ)

Case 1)

$$f''(t) = 0 \implies f(t) = c_1 + c_2 t \qquad \text{if } \lambda = 0 \qquad f(t) = (c_1 + c_2 t)\vec{v}$$

Case 2)

$$f(t) = c_1\cos(\omega t) + c_2\sin(\omega t) \qquad \text{if } \lambda < 0,\ \lambda = -\omega^2 \quad \omega = \sqrt{-\lambda}$$

$$f''(t) + \underbrace{(-\lambda)}_{>0} f(t) = 0$$

$$f''(t) + \omega^2 f(t) = 0$$

Case 3)

$$f(t) = c_1 e^{\sqrt{\lambda}\,t} + c_2\,e^{-\sqrt{\lambda}\,t} \qquad \text{if } \lambda > 0\,. \qquad f''(t) - \lambda f(t) = 0$$

Case 3 will never happen for our mass-spring configurations, because of conservation of energy!

This leads to the

Solution space algorithm: Consider a very special case of a homogeneous system of linear differential equations,

$$\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}} \; .$$

If $A_{n \times n}$ is a diagonalizable matrix and if all of its eigenvalues are non-positive then for each eigenpair $\left(\lambda_j, \underline{\mathbf{v}}_j\right)$ with $\lambda_j < 0$ there are two linearly independent sinusoidal solutions to $\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}$ given by

$$\mathbf{x}_j(t) = \cos\left(\omega_j\, t\right)\underline{\mathbf{v}}_j \qquad \underline{\mathbf{y}}_j(t) = \sin\left(\omega_j\, t\right)\underline{\mathbf{v}}_j$$

with

$$\omega_j = \sqrt{-\lambda_j} \; .$$

And for an eigenpair $\left(\lambda_j, \underline{\mathbf{v}}_j\right)$ with $\lambda_j = 0$ there are two indpendent solutions given by constant and linear functions

$$\mathbf{x}_j(t) = \underline{\mathbf{v}}_j \qquad \underline{\mathbf{y}}_j(t) = t\,\underline{\mathbf{v}}_j$$

This procedure constructs $2\,n$ independent solutions to the system $\underline{\mathbf{x}}''(t) = A\,\underline{\mathbf{x}}$, i.e. a basis for the solution space.

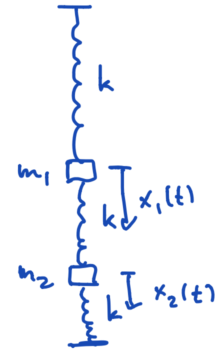
Remark: What's amazing is that the fact that if the system is conservative, the acceleration matrix will always be diagonalizable, and all of its eigenvalues will be non-positive. In fact, if the system is tethered to at least one wall (as in the first two diagrams on page 1), all of the eigenvalues will be strictly negative, and the algorithm above will always yield a basis for the solution space. (If the system is not tethered and is free to move as a train, like the third diagram on page 1, then $\lambda = 0$ will be one of the eigenvalues, and will yield the constant velocity and displacement contribution to the solution space, $\left(c_1 + c_2\, t\right)\underline{\mathbf{v}}$, where $\underline{\mathbf{v}}$ is the corresponding eigenvector. Together with the solutions from strictly negative eigenvalues this will still lead to the general homogeneous solution.)

Exercise 2) Consider the special case of the configuration on page one for which $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$. In this case, the equation for the vector of the two mass accelerations reduces to

$\vec{x}'' = A\vec{x}$ if

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{k}{m} \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$



a) Find the eigendata for the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

b) Deduce the eigendata for the acceleration matrix A which is $\frac{k}{m}$ times this matrix.

c) Find the 4 - dimensional solution space to this two-mass, three-spring system.

a). $\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 - 1 = (\lambda+3)(\lambda+1)$

$\lambda = -3, -1.$

$E_{\lambda=-1}$ $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$

$E_{\lambda=-1} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$E_{\lambda=-3}$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$E_{\lambda=-3} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}.$

b). If $A\vec{v} = \lambda\vec{v}$

what is $(sA)\vec{v} = s(A\vec{v}) = (s\lambda)\vec{v}$

if mult matrix by constant, eigenvectors stays same. eigenvalues get mult by const

for $\frac{k}{m}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$\lambda = -3\frac{k}{m}, -1\cdot\frac{k}{m}$

$\omega = \underbrace{\sqrt{\frac{3k}{m}}}_{=\,\omega_2}, \underbrace{\sqrt{\frac{k}{m}}}_{=\,\omega_1}$

c) soltns:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \underbrace{\left[c_1 \cos\sqrt{\tfrac{k}{m}}\,t + c_2 \sin\sqrt{\tfrac{k}{m}}\,t\right]}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{\left[c_3 \cos\left(\sqrt{\tfrac{3k}{m}}\,t\right) + c_4 \sin\left(\sqrt{\tfrac{3k}{m}}\,t\right)\right]}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

"slow" in-phase

amplitude - phase form

"fast" out of phase

<u>solution</u> The general solution is a superposition of two "fundamental modes". In the slower mode both masses oscillate "in phase", with equal amplitudes, and with angular frequency $\omega_1 = \sqrt{\frac{k}{m}}$. In the faster mode, both masses oscillate "out of phase" with equal amplitudes, and with angular frequency $\omega_2 = \sqrt{\frac{3k}{m}}$. The general solution can be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = C_1 \cos(\omega_1 t - \alpha_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \cos(\omega_2 t - \alpha_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3 \cos(\omega_2 t) + c_4 \sin(\omega_2 t)\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

<u>Exercise 3)</u> Show that the general solution above lets you uniquely solve each IVP uniquely. This should reinforce the idea that the solution space to these two second order linear homgeneous DE's is <u>four</u> dimensional.

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1(0) = a_1, \quad x_1'(0) = a_2$$
$$x_2(0) = b_1, \quad x_2'(0) = b_2$$

units for k $\quad N/m \quad kg/s^2$

$m \quad kg$

$$\frac{k}{m} = \frac{1}{s^2}$$