Theorem 1   For the IVP

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If $\underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}})$ is continuous in the $t-$variable and differentiable in its $\underline{\boldsymbol{x}}$ variable, then there exists a unique solution to the IVP, at least on some (possibly short) time interval $t_0 - \delta < t < t_0 + \delta$.

Theorem 2   For the special case of the first order linear system of differential equations IVP

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

Remark: The solutions to these systems of DE's may be approximated numerically using vectorized versions of Euler's method and the Runge Kutta method. The ideas are exactly the same as they were for scalar equations, except that they now use vectors. This is how commerical numerical DE solvers work. For example, with time-step h the Euler loop would increment as follows:

$$t_j = t_0 + h\,j$$
$$\underline{\boldsymbol{x}}_{j+1} = \underline{\boldsymbol{x}}_j + h\,\underline{\boldsymbol{F}}(t_j, \underline{\boldsymbol{x}}_j).$$

Remark: These theorems are the true explanation for why the n^{th}-order linear DE IVPs in Chapter 5 always have unique solutions - because each $n^{th} - order$ linear DE IVP is equivalent to an IVP for a first order system of n linear DE's. In fact, when software finds numerical approximations for solutions to higher order (linear or non-linear) DE IVPs that can't be found by the techniques of Chapter 5 or other mathematical formulas, it works by converting these IVPs to the equivalent first order system IVPs, and uses algorithms like Euler and Runge-Kutta to approximate the solutions.

Theorem 3) Vector space theory for first order systems of linear DEs (Notice the familiar themes...we can completely understand these facts if we take the intuitively reasonable existence-uniqueness Theorem 2 as fact.)

3.1) For vector functions $\underline{\boldsymbol{x}}(t)$ differentiable on an interval, the operator

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) - A(t)\underline{x}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{z}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{z}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t))\,.$$

check!

3.2) Thus, by the fundamental theorem for linear transformations, the general solution to the non-homogeneous linear problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

$\forall\ t \in I$ is

$$\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{x}}_p(t) + \underline{\boldsymbol{x}}_H(t)$$

where $\underline{\boldsymbol{x}}_p(t)$ is any single particular solution and $\underline{\boldsymbol{x}}_H(t)$ is the general solution to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$

We frequently write the homogeneous linear system of DE's as

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t)\,.$$

3.3) For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

is n-dimensional. Here's why:

- Let $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ be any n solutions to the homogeneous problem chosen so that the Wronskian matrix at $t_0 \in I$

$$[W(\underline{\boldsymbol{X}}_1, \underline{\boldsymbol{X}}_2, \dots, \underline{\boldsymbol{X}}_n)](t_0) := [\underline{\boldsymbol{X}}_1(t_0) | \underline{\boldsymbol{X}}_2(t_0) | \cdots | \underline{\boldsymbol{X}}_n(t_0)]$$

is invertible. (By the existence theorem we can choose solutions for any collection of initial vectors - so for example, in theory we could pick the matrix above to actually equal the identity matrix. In practice we'll be happy with any invertible matrix.)

- Then for any $\underline{\boldsymbol{b}} \in \mathbb{R}^n$ the IVP

$$\begin{cases} \underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}} \\ \underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{b}} \end{cases}$$

has solution $\underline{\boldsymbol{x}}(t) = c_1 \underline{\boldsymbol{X}}_1(t) + c_2 \underline{\boldsymbol{X}}_2(t) + \ldots + c_n \underline{\boldsymbol{X}}_n(t)$ where the linear combination coefficients are the solution to the Wronskian matrix equation

$$\vec{x}(t_0) = \left[\begin{array}{c|c|c|c} \underline{\boldsymbol{X}}_1(t_0) & \underline{\boldsymbol{X}}_2(t_0) & \cdots & \underline{\boldsymbol{X}}_n(t_0) \end{array}\right] \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_n \end{bmatrix}.$$

Thus, because the Wronskian matrix at t_0 is invertible, every IVP can be solved with a linear combination of $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$, and since each IVP has only one solution, $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ span the solution space. The same matrix equation shows that the only linear combination that yields the zero function (which has initial vector $\underline{\boldsymbol{b}} = \underline{\boldsymbol{0}}$) is the one with $\underline{\boldsymbol{c}} = \underline{\boldsymbol{0}}$. Thus $\underline{\boldsymbol{X}}_1(t), \underline{\boldsymbol{X}}_2(t), ...\underline{\boldsymbol{X}}_n(t)$ are also linearly independent. Therefore they are a basis for the solution space, and their number n is the dimension of the solution space.

7.3 Eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

Here's how: We look for a basis of solutions $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is a constant vector. Substituting this form of potential solution into the system of DE's above yields the equation

$$\lambda\, e^{\lambda t}\underline{\boldsymbol{v}} = A\, e^{\lambda t}\underline{\boldsymbol{v}} = e^{\lambda t} A\,\underline{\boldsymbol{v}}\,.$$

Dividing both sides of this equation by the scalar function $e^{\lambda t}$ gives the condition

$$\lambda\,\underline{\boldsymbol{v}} = A\,\underline{\boldsymbol{v}}\,.$$

- We get a solution every time $\underline{\boldsymbol{v}}$ is an eigenvector of A with eigenvalue λ !
- If A is diagonalizable then there is an $\mathbb{R}^n$ basis of eigenvectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ and solutions

$$\underline{\boldsymbol{X}}_1(t) = e^{\lambda_1 t}\underline{\boldsymbol{v}}_1,\ \underline{\boldsymbol{X}}_2(t) = e^{\lambda_2 t}\underline{\boldsymbol{v}}_2,\ \ldots,\ \underline{\boldsymbol{X}}_n(t) = e^{\lambda_n t}\underline{\boldsymbol{v}}_n$$

which are a basis for the solution space on the interval $I = \mathbb{R}$, because the Wronskian matrix at $t = 0$ is the invertible diagonalizing matrix

$$\boxed{P = [\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n]}$$

that we considered in Chapter 6.

- If A has complex number eigenvalues and eigenvectors it may still be diagonalizable over $\mathbb{C}^n$, and we will still be able to extract a basis of real vector function solutions. If A is not diagonalizable over $\mathbb{R}^n$ or over $\mathbb{C}^n$ the situation is more complicated.

Exercise 4a) Use the method above to find the general homogeneous solution to

$$* \quad \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4b) Solve the IVP with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}}$$

$$\vec{x}_H(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

eigendata for A:

① $|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = \lambda(7+\lambda) + 6 = \lambda^2 + 7\lambda + 6 = (\lambda+6)(\lambda+1)$

roots $\lambda = -1, -6$.

② $E_{\lambda=-1}$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -6 & -6 & 0 \end{array}\right]$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$1\cdot col_1 - 1\cdot col_2 = \vec{0}$

soln $e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$E_{\lambda=-6}$ $\left[\begin{array}{cc|c} 6 & 1 & 0 \\ -6 & -1 & 0 \end{array}\right]$

$\vec{v} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$

soln $e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}.$

solns to * are

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

@ $t=0$: IVP $\begin{bmatrix} 1 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -6 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -6 & 4 \end{array}\right]$

R_1+R_2 $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -5 & 5 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$

$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$

4 $\rightarrow \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$; $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$c_1 = 2$
$c_2 = -1$

Exercise 5a) What second order overdamped initial value problem is equivalent to the first order system IVP on the previous page. And what is the solution function to this IVP?

5b) What do you notice about the Chapter 5 "Wronskian matrix" for the second order DE in 4a, and the Chapter 7 "Wronskian matrix" for the solution to the equivalent first order system?

5c) Since in the correspondence above, $x_2(t)$ equals the mass velocity $x'(t) = v(t)$, I've created the pplane phase portrait below using the lettering $[x(t), v(t)]^T$ rather than $[x_1(t), x_2(t)]^T$. Intepret the behavior of the overdamped mass-spring motion in terms of the pplane phase portrait.

5d) How do the eigenvectors show up in the phase portrait, in terms of the direction the origin is approached from as $t \rightarrow \infty$, and the direction solutions came from (as $t \rightarrow -\infty$)?

http://math.rice.edu/~dfield/dfpp.html

5a $x_1' = x_2$

$x_1'' = x_2' = -6x_1 - 7\underbrace{x_2}_{x_1'}$

$x_1'' = -6x_1 - 7x_1'$

$x_1'' + 7x_1' + 6x_1 = 0$

so $x_1(t)$ solves

$\begin{cases} x'' + 7x' + 6x = 0 \\ x(0) = 1 \\ x'(0) = 4. \end{cases}$

$x_1(t) = 2e^{-t} - 1e^{-6t}$

from previous page is the soln

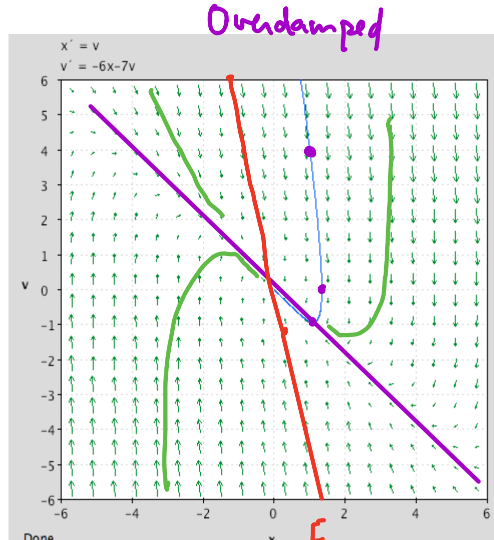
check: $x_1(0) = 1$

$x_1'(0) = -2 + 6 = 4$ ✓

$(= x_2(0))$

5c $x(0) = 1$
$x'(0) = 4$

Overdamped



$E_{\lambda=-6}$ $E_{\lambda=-1}$

5b W from #4

$\begin{bmatrix} e^{-t} & e^{-6t} \\ -e^{-t} & -6e^{-6t} \end{bmatrix}$

Same!

5d phase diagram & eigendata

$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t}\begin{bmatrix} 1 \\ -6 \end{bmatrix}$

as $t \rightarrow \infty$
$\vec{x}(t) \rightarrow \vec{0}$
approaches $\vec{0}$
tangent to $E_{\lambda=-1}$
(unless $c_1 = 0$)

$\lambda = -1$
$E_{\lambda=-1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$\lambda = -6$
$E_{\lambda=-6}$
$= span\left\{\begin{bmatrix} 1 \\ -6 \end{bmatrix}\right\}$

as $t \rightarrow -\infty$
tangent directions are parallel to $E_{\lambda=-6}$
(unless $c_2 = 0$)

• postpone 7.3.34, w13.3 until next HW.
• 1^{st} page of today's notes
• Finish Monday's notes

Math 2250-004
Tues Apr 18
7.1-7.3 Summary of what is covered in Monday's and Tuesday's notes:

• Any initial value problem for a differential equation or system of differential equations can be converted into an equivalent initial value problem for a system of first order differential equations.

• There is a "short-time" existence-uniqueness theorem for first order DE initial value problems

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

and solutions can be approximated using Euler or Runge-Kutta type algorithms. •

• • A special case of the IVP above is the one for a first order linear system of differential equations :

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

If the matrix $A(t)$ and the vector function $\underline{\boldsymbol{f}}(t)$ are continuous on an open interval I containing t_0 then a solution $\underline{\boldsymbol{x}}(t)$ exists and is unique, on the entire interval.

• The general solution to an inhomogeneous linear system of DEs

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{f}}(t)$$
i.e.
$$L(\vec{x}(t)) = \underline{\boldsymbol{x}}'(t) - A(t)\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{f}}(t)$$

will be of the form $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$ where $\underline{\boldsymbol{x}}_P$ is a particular solution, and $\underline{\boldsymbol{x}}_H$ is the general solution to the homogeneous system

$$\underline{\boldsymbol{x}}'(t) = A(t)\underline{\boldsymbol{x}}(t) .$$

• For $A(t)_{n \times n}$ and $\underline{\boldsymbol{x}}(t) \in \mathbb{R}^n$ the solution space on the $t-$interval I to the homogeneous problem

$$\underline{\boldsymbol{x}}'(t) = A(t)\,\underline{\boldsymbol{x}}(t)$$
i.e.
$$\underline{\boldsymbol{x}}'(t) - A(t)\,\underline{\boldsymbol{x}}(t) = \underline{\boldsymbol{0}}$$

is n-dimensional.

• For $A_{n \times n}$ a constant matrix, we try to find a basis for the solution space to

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

consisting of solutions of the form $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$, where $\underline{\boldsymbol{v}}$ is an eigenvector of A with eigenvalue λ. We will succeed as long as A is diagonalizable.

Today: We will continue using the eigenvalue-eigenvector method for finding the general solution to the homogeneous constant matrix first order system of differential equations

$$\underline{\boldsymbol{x}}' = A\,\underline{\boldsymbol{x}}$$

that we discussed yesterday, and is in Monday's notes. Today we'll consider examples where the eigenvalues and eigenvectors are complex. There is such an example in the homework due Wednesday.... problem w13.3.

So far we've not considered the possibility of complex eigenvalues and eigenvectors. Linear algebra theory works the same with complex number scalars and vectors - one can talk about complex vector spaces, linear combinations, span, linear independence, reduced row echelon form, determinant, dimension, basis, etc. Then the model space is $\mathbb{C}^n$ rather than $\mathbb{R}^n$.

Definition: $\underline{\boldsymbol{v}} \in \mathbb{C}^n$ ($\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$) is a complex eigenvector of the matrix A, with eigenvalue $\lambda \in \mathbb{C}$ if $A\,\underline{\boldsymbol{v}} = \boldsymbol{\lambda}\,\underline{\boldsymbol{v}}$.

Just as before, you find the possibly complex eigenvalues by finding the roots of the characteristic polynomial $|A - \lambda I|$. Then find the eigenspace bases by reducing the corresponding matrix (using complex scalars in the elementary row operations).

The best way to see how to proceed in the case of complex eigenvalues/eigenvectors is to work an example. There is a general discussion on the page after this example that we will refer to along the way:

Glucose-insulin model (adapted from a discussion on page 339 of the text "Linear Algebra with Applications," by Otto Bretscher)

Let $G(t)$ be the excess glucose concentration (mg of G per 100 ml of blood, say) in someone's blood, at time t hours. Excess means we are keeping track of the difference between current and equilibrium ("fasting") concentrations. Similarly, Let $H(t)$ be the excess insulin concentration at time t hours. When blood levels of glucose rise, say as food is digested, the pancreas reacts by secreting insulin in order to utilize the glucose. Researchers have developed mathematical models for the glucose regulatory system. Here is a simplified (linearized) version of one such model, with particular representative matrix coefficients. It would be meant to apply between meals, when no additional glucose is being added to the system:

$$\begin{bmatrix} G'(t) \\ H'(t) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.4 \\ 0.1 & -0.1 \end{bmatrix} \begin{bmatrix} G \\ H \end{bmatrix}$$

$G' = -.1G - .4H$
$H' = .1G - .1H$

Exercise 3a) Understand why the signs of the matrix entries are reasonable.

Now let's solve the initial value problem, say right after a big meal, when

$$\begin{bmatrix} G(0) \\ H(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

3b) The first step is to get the eigendata of the matrix. Do this, and compare with the Maple check on the next page.

$$\begin{vmatrix} -.1-\lambda & -.4 \\ .1 & -.1-\lambda \end{vmatrix} = (\lambda+.1)^2 + .04 = 0$$

$(\lambda+.1)^2 = -.04$

$\lambda + .1 = \pm .2i$

$\lambda = -.1 \pm .2i$

$E_{\lambda=-.1+.2i}$: $\left[\begin{array}{cc|c} -.2i & -.4 & 0 \\ .1 & -.2i & 0 \end{array}\right]$

$\begin{array}{l} -5R_1 \\ 10R_2 \end{array} \left[\begin{array}{cc|c} i & 2 & 0 \\ 1 & -2i & 0 \end{array}\right]$ $\leftarrow R_1 = iR_2$ ✓