In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{x}) = A\,\underline{x}$$

almost as well as if A is a diagonal matrix, and so we call such matrices diagonalizable. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word diagonalizable to describe such matrices.

Use the $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in Exercise 1, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column:

basis for $E_{\lambda=2}$ basis for $E_{\lambda=3}$

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

B

Now do algebra (check these steps and discuss what's going on!)

BP $B\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}\left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{array}\right]$$

diagonal matrix of evals

$$= \left[\begin{array}{c|c|c} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{array}\right]\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad = PD$$

In other words,

$$B\,P = P\,\mathrm{D},$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P). Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P\,\mathrm{D}\,P^{-1} \text{ and } P^{-1}BP = \mathrm{D}.$$

Exercise 2) Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

$$B^2 = (PD\underbrace{P^{-1})(P}_{I}DP^{-1}) = PD^2P^{-1} = P\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}P^{-1}$$

$$B^{100} = PD\cancel{P^{-1}P}D\cancel{P^{-1}P}DP^{-1}\cdots PDP^{-1}$$

$$= PD^{100}P^{-1}$$

Definition: Let $A_{n\times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 | \lambda_2\underline{\boldsymbol{v}}_2 | \cdots | \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 | \underline{\boldsymbol{v}}_2 | \cdots | \underline{\boldsymbol{v}}_n\right]\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ : & : & \ldots & : \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, not all matrices are diagonalizable:

Exercise 3) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable.

① $|C-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)^2(3-\lambda) = -(\lambda-2)^2(\lambda-3) = 0$

$\lambda = 2, 3$

$E_{\lambda=2}$: $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\vec{v} = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$1\cdot col_1 + 0\, col_2 + 0\, col_3 = \vec{0}$

or backsolve:

$v_2 = 0$

$0 = 0$

$v_3 = 0$

& $v_1 = t$.

$E_{\lambda=2} = span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

$E_{\lambda=3}$: $\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$E_{\lambda=3} = span\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

cannot make a basis of $\mathbb{R}^3$ out of eigenvectors.

Facts about diagonalizability (see text section 6.2 for complete discussion, with reasoning):

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \dots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \dots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \leq dim\left(E_{\lambda = \lambda_j}\right) \leq k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called defective.
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

- In the special case that A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ each eigenspace is forced to be

1-dimensional since $k_1 + k_2 + \dots + k_n = n$ so each $k_j = 1$. Thus A is automatically diagonalizable as a special case of the second bullet point.

Exercise 4) How do the examples from today and yesterday compare with the general facts about diagonalizability?

#5 Tuesday: $B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

on Wed

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

$E_{\lambda=2} = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$, $E_{\lambda=3} = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

#3 Wed: $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

on Fri

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

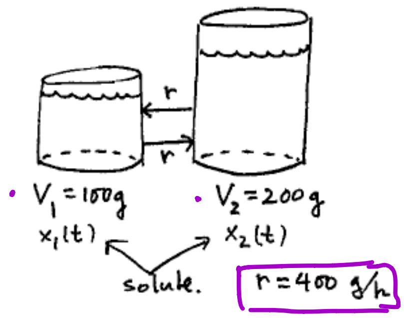
$E_{\lambda=2} = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $E_{\lambda=3} = span\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

• check off on diagonalizability
• do today's notes

Math 2250-004
Fri Apr 14

7.1 Systems of differential equations - to model multi-component systems via compartmental analysis:
http://en.wikipedia.org/wiki/Multi-compartment_model

Here's a relatively simple 2-tank problem to illustrate the ideas:



Exercise 1) Find differential equations for solute amounts $x_1(t), x_2(t)$ above, using input-output modeling. Assume solute concentration is uniform in each tank. If $x_1(0) = b_1, x_2(0) = b_2$, write down the initial value problem that you expect would have a unique solution.

$x_1'(t) = r_i \cdot c_i - r_o c_o = (400)\left(\frac{x_2}{200}\right) - 400\left(\frac{x_1}{100}\right) = -4x_1 + 2x_2$

(r_i: ~~vol~~/time, c_i: mass/~~vol~~)

$x_2'(t) = r_i c_i - r_o c_o = 400\frac{x_1}{100} - 400\frac{x_2}{200} = 4x_1 - 2x_2.$

IVP
$$\begin{cases} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \end{cases}$$

answer (in matrix-vector form):

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Geometric interpretation of first order systems of differential equations.

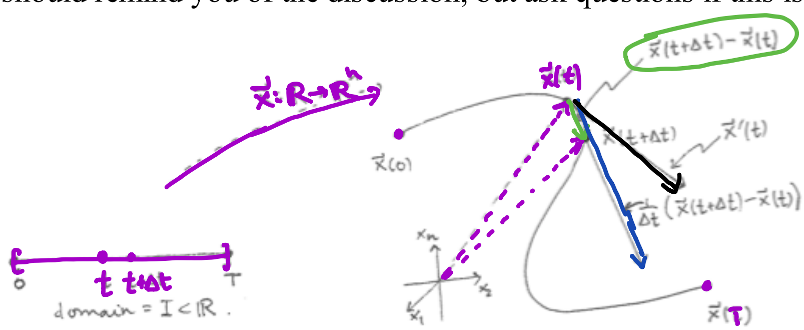
The example on page 1 is a special case of the general initial value problem for a first order system of differential equations:

$$\underline{\boldsymbol{x}}'(t) = \underline{\boldsymbol{F}}(t, \underline{\boldsymbol{x}}(t))$$
$$\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$$

• We will see how any single differential equation (of any order), or any system of differential equations (of any order) is equivalent to a larger first order system of differential equations. And we will discuss how the natural initial value problems correspond.

Why we expect IVP's for first order systems of DE's to have unique solutions $\underline{\boldsymbol{x}}(t)$:

• From either a multivariable calculus course, or from physics, recall the geometric/physical interpretation of $\underline{\boldsymbol{x}}'(t)$ as the tangent/velocity vector to the parametric curve of points with position vector $\underline{\boldsymbol{x}}(t)$, as t varies. This picture should remind you of the discussion, but ask questions if this is new to you:



Analytically, the reason that the vector of derivatives $\underline{\boldsymbol{x}}'(t)$ computed component by component is actually a limit of scaled secant vectors (and therefore a tangent/velocity vector) is:

$$\underline{\boldsymbol{x}}'(t) := \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\begin{bmatrix} x_1(t+\Delta t) \\ x_2(t+\Delta t) \\ \vdots \\ x_n(t+\Delta t) \end{bmatrix} - \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}\right)$$

$$= \lim_{\Delta t \to 0} \begin{bmatrix} \frac{1}{\Delta t}\left(x_1(t+\Delta t) - x_1(t)\right) \\ \frac{1}{\Delta t}\left(x_2(t+\Delta t) - x_2(t)\right) \\ \vdots \\ \frac{1}{\Delta t}\left(x_n(t+\Delta t) - x_n(t)\right) \end{bmatrix} = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix},$$

provided each component function is differentiable. Therefore, the reason you expect a unique solution to the IVP for a first order system is that you know where you start ($\underline{\boldsymbol{x}}(t_0) = \underline{\boldsymbol{x}}_0$), and you know your "velocity" vector (depending on time and current location) $\Rightarrow$ you expect a unique solution! (Plus, you could use something like a vector version of Euler's method or the Runge-Kutta method to approximate it! You just convert the scalar quantities in the code into vector quantities. And this is what numerical solvers do.)

Exercise 2) Return to the page 1 tank example

IVP
$$\begin{cases} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4x_1 + 2x_2 \\ 4x_1 - 2x_2 \end{bmatrix} = (4x_1 - 2x_2)\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ x_1(0) = 9 \quad \leftarrow \text{9 lbs start in 1st tank} \\ x_2(0) = 0 \quad \leftarrow \text{0 '' '' '' 2nd tank.} \end{cases}$$

2a) Interpret the parametric solution curve $[x_1(t), x_2(t)]^T$ to this IVP, as indicated in the pplane screen shot below. ("pplane" is the sister program to "dfield", that we were using in Chapters 1-2.) Notice how it follows the "velocity" vector field (which is time-independent in this example), and how the "particle motion" location $[x_1(t), x_2(t)]^T$ is actually the vector of solute amounts in each tank, at time *t*. If your system involved ten coupled tanks rather than two, then this "particle" is moving around in $\mathbb{R}^{10}$.

2b) What are the apparent limiting solute amounts in each tank?

2c) How could your smart-alec younger sibling have told you the answer to 2b without considering any differential equations or "velocity vector fields" at all?

2b $\lim_{t \to \infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$V_1 = 100$ gal.

$V_2 = 200$ gal.

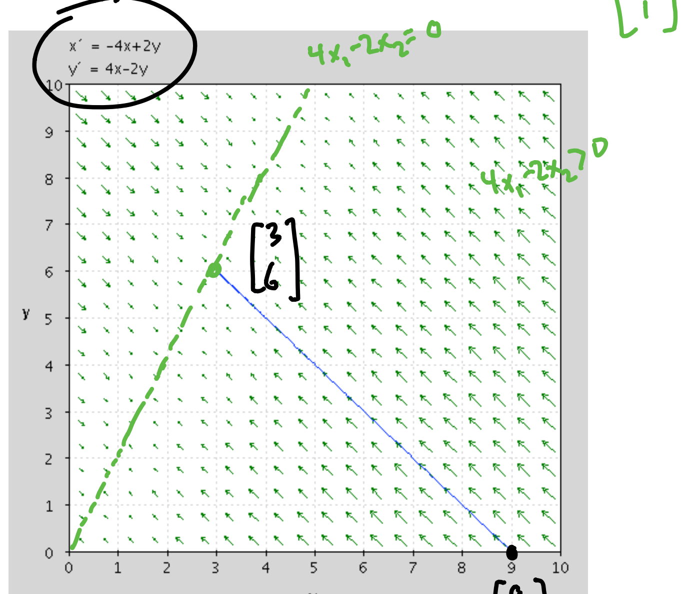
2c limiting concentration should be the same in each tank,

so twice as much in 2nd tank & 9 lbs total.

Duh.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4x_1 - 2x_2 \\ 4x_1 - 2x_2 \end{bmatrix}$$

$$= (4x_1 - 2x_2)\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



$\begin{bmatrix} 9 \\ 0 \end{bmatrix}$ initial point

Chapter 1.

First order systems of differential equations of the form

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$

are called linear homogeneous systems of DE's. (Think of rewriting the system as

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

$x'(t) + p(t)\,x(t) = 0$

in analogy with how we wrote linear scalar differential equations.) Then the inhomogeneous system of first order DE's would be written as

$$\boxed{\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)}$$

$x'(t) + p(t)\,x(t) = q(t)$

or

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}} + \underline{\boldsymbol{f}}(t)$$

Notice that the operator on vector-valued functions $\underline{\boldsymbol{x}}(t)$ defined by

$$L(\underline{\boldsymbol{x}}(t)) := \underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}}(t)$$

is linear, i.e.

$$L(\underline{\boldsymbol{x}}(t) + \underline{\boldsymbol{y}}(t)) = L(\underline{\boldsymbol{x}}(t)) + L(\underline{\boldsymbol{y}}(t))$$
$$L(c\,\underline{\boldsymbol{x}}(t)) = c\,L(\underline{\boldsymbol{x}}(t)).$$

$$\begin{aligned} L(\vec{x}+\vec{y}) &= (\vec{x}+\vec{y})' - A(\vec{x}+\vec{y}) \\ &= \vec{x}' + \vec{y}' - A\vec{x} - A\vec{y} \\ &= (\vec{x}' - A\vec{x}) + (\vec{y}' - A\vec{y}) \\ &= L(\vec{x}) + L(\vec{y}) \end{aligned}$$

SO! The space of solutions to the homogeneous first order system of differential equations

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$$

is a subspace. AND the general solution to the inhomogeneous system

$$\underline{\boldsymbol{x}}'(t) - A\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{f}}(t)$$

will be of the form

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}_P + \underline{\boldsymbol{x}}_H$$

where $\underline{\boldsymbol{x}}_P$ is any single particular solution and $\underline{\boldsymbol{x}}_H$ is the general homogeneous solution.

Exercise 3) In the case that A is a constant matrix (i.e. entries don't depend on t), consider the homogeneous problem

$$* \qquad \underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}\,.$$

Look for solutions of the form

$$\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}\,,$$

where $\underline{\boldsymbol{v}}$ is a constant vector. Show that $\underline{\boldsymbol{x}}(t) = e^{\lambda t}\underline{\boldsymbol{v}}$ solves the homogeneous DE system if and only if v is an eigenvector of A, with eigenvalue λ, i.e. $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$.

Hint: In order for such an $\underline{\boldsymbol{x}}(t)$ to solve the DE it must be true that

LHS (*) $\qquad \underline{\boldsymbol{x}}'(t) = \lambda\,\cancel{e^{\lambda t}}\,\underline{\boldsymbol{v}}$

||

and

RHS (*) $\qquad A\,\underline{\boldsymbol{x}}(t) = A\,e^{\lambda t}\underline{\boldsymbol{v}} = \cancel{e^{\lambda t}}\,A\,\underline{\boldsymbol{v}}$

$= \text{ if } \underline{A\vec{v} = \lambda\vec{v}}.$

Set these two expressions equal.

basis of solns for $\vec{x}' = A\vec{x}$ of form $e^{\lambda t}\vec{v}$.

Exercise 4) Use the idea of Exercise 3 to solve the initial value problem of Exercise 2!! Compare your solution $\underline{\boldsymbol{x}}(t)$ to the parametric curve drawn by pplane, that we looked at a couple of pages back.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$$

(A) get eigendata for A

(1) $\begin{vmatrix} -4-\lambda & 2 \\ 4 & -2-\lambda \end{vmatrix} = (\lambda+4)(\lambda+2) - 8$
$= \lambda^2 + 6\lambda + \cancel{8} \cancel{-8}$
$= \lambda(\lambda+6)$

evals $\lambda = 0, -6$.

$E_{\lambda=0}$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$E_{\lambda=0} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \xrightarrow{\text{soln}} e^{\lambda t}\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$1 \cdot \text{col}_1 + 2\,\text{col}_2 = 0$

$E_{\lambda=-6}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right]$$

$E_{\lambda=-6} = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \xrightarrow{\text{soln}} e^{-6t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

IVP $\begin{bmatrix} 9 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ @ $t=0$

$9 = c_1 + c_2$
$0 = 2c_1 - c_2$

$c_1 = 3$
$c_2 = 6$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6e^{-6t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Exercise 5) Lessons learned from tank example: What condition on the matrix $A_{n \times n}$ will allow you to uniquely solve every initial value problem

$$\underline{\boldsymbol{x}}'(t) = A\,\underline{\boldsymbol{x}}$$
$$\underline{\boldsymbol{x}}(0) = \underline{\boldsymbol{x}}_0 \in \mathbb{R}^n$$

using the method in Exercise 3-4 ? Hint: Chapter 6. (If that condition fails there are other ways to find the unique solutions.)