Exercise 4) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 3.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

solve $(A-\lambda I)\vec{v} = \vec{0}$
- find λ's
- each λ, find $\vec{v}$'s.

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$T(c_1\vec{v} + c_2\vec{u}) = c_1 T(\vec{v}) + c_2 T(\vec{u}) = c_1 4\vec{v} + c_2\vec{u}$$

(1) $$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 2 = (\lambda-3)(\lambda-2) - 2 = \lambda^2 - 5\lambda + 6 - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1) = 0$$

e.values are $\lambda = 1, 4$.

(2a) $E_{\lambda=4}$ $\quad (A-4I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$v_1 = 2t$
$v_2 = t$

$\vec{v} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $\quad E_{\lambda=4} = span\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ $\quad \vec{v}$

Shortcut: homog solns correspond to column dependencies

$2\,col_1 + col_2 = \vec{0}$ so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis!

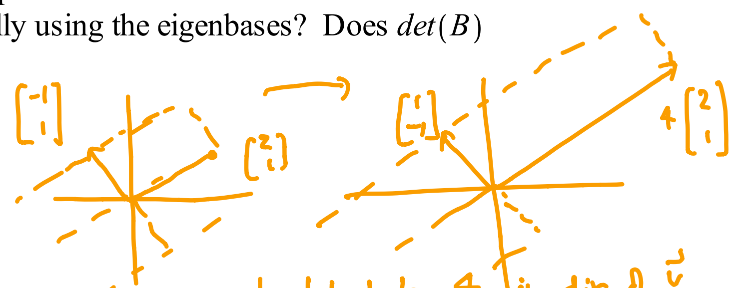
(2b) $E_{\lambda=1}$ $\quad \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 1 & 1 & 0 \end{array}\right]$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\quad E_{\lambda=1} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ $\quad \vec{u}$

Exercise 5) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

(i) Find the characteristic polynomial and factor it to find the eigenvalues.
(ii) for each eigenvalue, find bases for the corresponding eigenspaces.
(iii) Can you describe the transformation $T(\underline{x}) = B\underline{x}$ geometrically using the eigenbases? Does $det(B)$ have anything to do with the geometry of this transformation?



stretched by 4 in dir. of $\vec{v}$
& by 1 in dir of $\vec{u}$

area expanded by 4.

$$\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 6 - 2 = 4 \checkmark$$

(1) Find eigenvalues

$$p(\lambda) = det(B - \lambda I) = 0$$

$$= \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{vmatrix}$$

expand accross to row, e.g.

$$\begin{array}{r} \\ \\ -R_2+R_3 \end{array} = \begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 0 & \lambda-2 & 2-\lambda \end{vmatrix}$$

$$=(\lambda-2)\begin{vmatrix} 4-\lambda & -2 & 1 \\ 2 & -\lambda & 1 \\ 0 & 1 & -1 \end{vmatrix} = (\lambda-2)\left(0 - 1\begin{vmatrix} 4-\lambda & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 4-\lambda & -2 \\ 2 & -\lambda \end{vmatrix}\right)$$

$$= (\lambda-2)\Big(-(4-\lambda-2) - \big(\underbrace{(4-\lambda)(-\lambda)+4}_{-(\lambda^2-4\lambda+4)}\big)\Big)$$

$$= (\lambda-2)(-\lambda^2+5\lambda-6)$$

$$= (\lambda-2)(-)(\underbrace{\lambda^2-5\lambda+6}_{(\lambda-2)(\lambda-3)}) = -(\lambda-2)^2(\lambda-3) \;!$$

roots $\lambda = 2, 3$.

② find eigenspaces.

$E_{\lambda=2}$:
$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right]$$

$R_1/2$
$$\left[\begin{array}{ccc|c} 1 & -1 & .5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

backsolve
$v_1 = r - .5t$
$v_2 = r$
$v_3 = t$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = r\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -.5 \\ 0 \\ 1 \end{bmatrix}$$

$$E_{\lambda=2} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

shortcut: column dependencies

$1\cdot col_1 + 1\cdot col_2 + 0\cdot col_3 = \vec{0}$ $\quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is eigenvectr.

$-col_1 + 0\, col_2 + 2 col_3 = \vec{0}$ $\quad \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ is an eigenvectr.

or $col_2 + 2 col_3 = \vec{0}$
$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ is evectr

$E_{\lambda=3}$
$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right]$$

$1\cdot col_1 + 1\cdot col_2 + 1\cdot col_3 = \vec{0}$

so $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvectr.

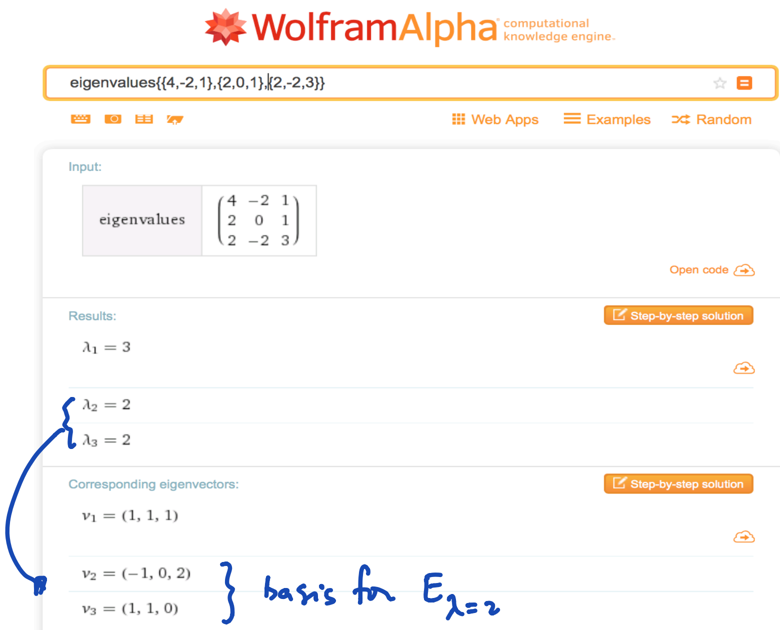
$\dim E_{\lambda=3} = 1$

$E_{\lambda=3} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (only 1 free param)

it turns out

$$\Big\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}}_{\substack{\lambda=2 \\ \text{evects}}}, \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\substack{\lambda=3 \\ \text{vects}}} \Big\}$$ are a basis for $\mathbb{R}^3$

Your solution will be related to the output below:



In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the geometry of the transformation

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$$

almost as well as if A is a diagonal matrix. This is actually something that does not always happen for a matrix A. When it does happen, we say that A is diagonalizable. Here's an example of a matrix which is NOT diagonalizable:

Exercise 7: Find matrix eigenvalues and eigenspace basis for each eigenvalue, for

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}.$$

Explain why there is no basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

~~skip~~

(1) $\begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 = 0$

$\lambda = 3$ only eigenvalue

(2) $E_{\lambda=3}$ $\quad \left[\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

backsolve.

$v_1 = t$

$v_2 = 0$!

$\vec{v} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$E_{\lambda=3} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

or

$1 \cdot col_1 + 0 \cdot col_2 = \vec{0}$

so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector

- pick up your old HW (& exams!)
- hand in HW due today
- in class quiz
- Lab tomorrow: eigenvalues - eigenvectors & what they have to do with DE's.
- next Thursday: final lab assignment.
  (final lab & final HW, due Tuesday @ 6 p.m.)

Math 2250-004
Wed Apr 12
6.1-6.2 Eigenvalues and eigenvectors for square matrices; diagonalizability of matrices.

Recall from yesterday,

Definition: If $A_{n \times n}$ and if $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ for some scalar λ and vector $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ then $\underline{\boldsymbol{v}}$ is called an eigenvector of A, and λ is called the eigenvalue of $\underline{\boldsymbol{v}}$ (and an eigenvalue of A).

- For general matrices, the eigenvector equation $A\,\underline{\boldsymbol{v}} = \lambda\,\underline{\boldsymbol{v}}$ can be rewritten as

$$(A - \lambda I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \, .$$

The only way such an equation can hold for $\underline{\boldsymbol{v}} \neq \underline{\boldsymbol{0}}$ is if the matrix $(A - \lambda I)$ does not reduce to the identity matrix. In other words - $det(A - \lambda I)$ must equal zero. Thus the only possible eigenvalues associated to a given matrix must be roots λ_j of the characteristic polynomial

1 $$p(\lambda) = det(A - \lambda I) \, .$$ set $p(\lambda)=0$ find roots λ_j

- So, the first step in finding eigenvectors for A is actually to find the eigenvalues - by finding the characteristic polynomial and its roots λ_j .

- For each root λ_j the matrix $A - \lambda_j I$ will not reduce to the identity, and the solution space to

2 solve $$(A - \lambda_j I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}}$$

will be at least one-dimensional, and have a basis of one or more eigenvectors. Find such a basis for this λ_j eigenspace $E_{\lambda = \lambda_j}$ by reducing the homogeneous matrix equation

$$(A - \lambda_j I)\underline{\boldsymbol{v}} = \underline{\boldsymbol{0}} \, ,$$

backsolving, and extracting a basis. We can often "see" and eigenvector by realizing that homogeneous solutions to a matrix equation correspond to column dependencies.

- Finish any leftover exercises from Tuesday

Exercise 1) If your matrix A is diagonal, the general algorithm for finding eigenspace bases just reproduces the entries along the diagonal as eigenvalues, and the corresponding standard basis vectors as eigenspace bases. (Recall our diagonal matrix examples from yesterday, where the standard basis vectors were eigenvectors. This is typical for diagonal matrices.) Illustrate how this works for a 3×3 diagonal matrix, so that in the future you can just read of the eigendata if the matrix you're given is (already) diagonal:

$$A := \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & 0 & 0 \\ 0 & a_{22}^2 & 0 \\ 0 & 0 & a_{33}^2 \end{bmatrix}$$

step 1) Find the roots of the characteristic polynomial $det(A - \lambda I)$.
step 2) Find the eigenspace bases, assuming the values of a_{11}, a_{22}, a_{33} are distinct (all different). What if $a_{11} = a_{22}$ but these values do not equal a_{33} ?

1) $$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & 0 & 0 \\ 0 & a_{22}-\lambda & 0 \\ 0 & 0 & a_{33}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$$

evals $\lambda = a_{11}, a_{22}, a_{33}$

2) $$A\vec{e}_1 = A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ 0 \\ 0 \end{bmatrix} = a_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ $\vec{e}_1$ evect, eval a_{11}

$$A\vec{e}_2 = a_{22}\vec{e}_2$$
$$A\vec{e}_3 = a_{33}\vec{e}_3 .$$

In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix $A_{n \times n}$, and putting them together, we get a basis for $\mathbb{R}^n$. This lets us understand the <u>geometry</u> of the transformation

$$T(\underline{\boldsymbol{x}}) = A\,\underline{\boldsymbol{x}}$$

almost as well as if A is a diagonal matrix, and so we call such matrices <u>diagonalizable</u>. Having such a basis of eigenvectors for a given matrix is also extremely useful for <u>algebraic</u> computations, and will give another reason for the word <u>diagonalizable</u> to describe such matrices.

Use the $\mathbb{R}^3$ basis made of out eigenvectors of the matrix B in <u>Exercise 1</u>, and put them into the columns of a matrix we will call P. We could order the eigenvectors however we want, but we'll put the $E_{\lambda=2}$ basis vectors in the first two columns, and the $E_{\lambda=3}$ basis vector in the third column:

basis for $E_{\lambda=2}$

basis for $E_{\lambda=3}$

B

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

Now do algebra (check these steps and discuss what's going on!)

BP $\qquad\qquad B\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{bmatrix}$$

diagonal matrix of evals

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \quad = PD$$

In other words,

$$B\,P = P\,\mathrm{D}\,,$$

where D is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in P).
Equivalently (multiply on the right by P^{-1} or on the left by P^{-1}):

$$B = P\,\mathrm{D}\,P^{-1} \text{ and } P^{-1}BP = \mathrm{D}.$$

<u>Exercise 2)</u> Use one of the the identities above to show how B^{100} can be computed with only two matrix multiplications!

$$B^2 = (PD\underbrace{P^{-1})(P}_{I}DP^{-1}) = PD^2P^{-1} = P\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}P^{-1}$$

$$B^{100} = \underbrace{PD\cancel{P^{-1}}}\underbrace{\cancel{P}D\cancel{P^{-1}}}\underbrace{\cancel{P}DP^{-1}}\cdots\underbrace{PDP^{-1}}$$

$$= PD^{100}P^{-1}$$

Definition: Let $A_{n \times n}$. If there is an $\mathbb{R}^n$ (or $\mathbb{C}^n$) basis $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots, \underline{\boldsymbol{v}}_n$ consisting of eigenvectors of A, then A is called diagonalizable. This is precisely why:

Write $A\,\underline{\boldsymbol{v}}_j = \lambda_j\,\underline{\boldsymbol{v}}_j$ (some of these λ_j may be the same, as in the previous example). Let P be the matrix

$$P = \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right].$$

Then, using the various ways of understanding matrix multiplication, we see

$$A\,P = A\left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] = \left[\lambda_1\underline{\boldsymbol{v}}_1 \middle| \lambda_2\underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \lambda_n\underline{\boldsymbol{v}}_n\right]$$

$$= \left[\underline{\boldsymbol{v}}_1 \middle| \underline{\boldsymbol{v}}_2 \middle| \cdots \middle| \underline{\boldsymbol{v}}_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

$$A\,P = P\,\mathrm{D}$$

$$A = P\,\mathrm{D}\,P^{-1}$$

$$P^{-1}A\,P = \mathrm{D}\,.$$

Unfortunately, not all matrices are diagonalizable:

Exercise 3) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable.

Facts about diagonalizability (see text section 6.2 for complete discussion, with reasoning):

Let $A_{n \times n}$ have factored characteristic polynomial

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \dots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each λ_j is distinct (i.e different). Notice that

$$k_1 + k_2 + \dots + k_m = n$$

because the degree of $p(\lambda)$ is n.

- Then $1 \le dim\left(E_{\lambda = \lambda_j}\right) \le k_j$. If $dim\left(E_{\lambda = \lambda_j}\right) < k_j$ then the λ_j eigenspace is called defective.
- The matrix A is diagonalizable if and only if each $dim\left(E_{\lambda = \lambda_j}\right) = k_j$. In this case, one obtains an $\mathbb{R}^n$

eigenbasis simply by combining bases for each eigenspace into one collection of n vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of $\mathbb{C}^n$.)

- In the special case that A has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ each eigenspace is forced to be

1-dimensional since $k_1 + k_2 + \dots + k_n = n$ so each $k_j = 1$. Thus A is automatically diagonalizable as a special case of the second bullet point.

Exercise 4) How do the examples from today and yesterday compare with the general facts about diagonalizability?

#5 Tuesday: $B = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

$E_{\lambda=2} = span\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$, $E_{\lambda=3} = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

#3 Wed: $C = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$p(\lambda) = -(\lambda-2)^2(\lambda-3)$

$E_{\lambda=2} = span\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$, $E_{\lambda=3} = span\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.