

Fri • on-off example in wed notes.  
 • impulse forcing & delta fun in today's notes

Monday • convolutions

Math 2250-4

Fri Apr 7

10.5, EP7.6 *continuing*

Today we ~~finish discussing~~ Laplace transform techniques:

- Impulse forcing ("delta functions")...today's notes.
- Convolution formulas to solve any inhomogeneous constant coefficient linear DE, with applications to interesting forced oscillation problems...today's notes.

Laplace table entries for today:

$f(t)$ with $ f(t)  \leq Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st} dt$ for $s > M$	comments
$u(t-a)$ unit step function	$\frac{e^{-as}}{s}$	for turning components on and off at $t=a$ .
$f(t-a)u(t-a)$	$e^{-as}F(s)$	more complicated on/off
$\delta(t-a)$	$e^{-as}$	unit impulse/delta "function"
$\int_0^t f(\tau)g(t-\tau) d\tau$	<i>Monday</i> $F(s)G(s)$	convolution integrals to invert Laplace transform products

EP 7.6 impulse functions and the  $\delta$  operator.

Consider a force  $f(t)$  acting on an object for only on a very short time interval  $a \leq t \leq a + \epsilon$ , for example as when a bat hits a ball. This impulse  $p$  of the force is defined to be the integral

$$p := \int_a^{a+\epsilon} f(t) dt$$

and it measures the net change in momentum of the object since by Newton's second law

$$\begin{aligned}
 m v'(t) &= f(t) \\
 \Rightarrow \int_a^{a+\epsilon} m v'(t) dt &= \int_a^{a+\epsilon} f(t) dt = p \\
 \Rightarrow m v(t) \Big|_{t=a}^{a+\epsilon} &= p.
 \end{aligned}$$

Since the impulse  $p$  only depends on the integral of  $f(t)$ , and since the exact form of  $f$  is unlikely to be known in any case, the easiest model is to replace  $f$  with a constant force having the same total impulse, i.e. to set

$$f = p d_{a,\epsilon}(t)$$

where  $d_{a,\epsilon}(t)$  is the unit impulse function given by

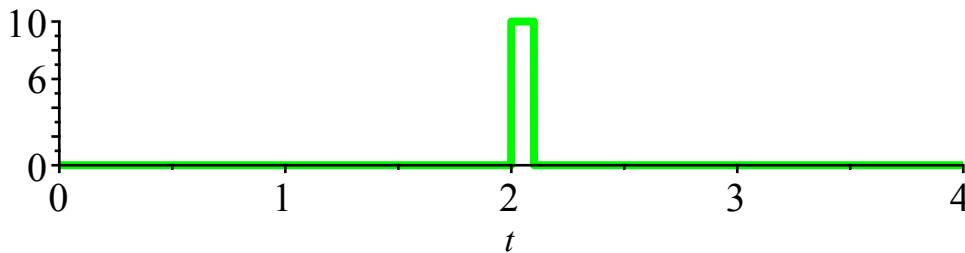
$$d_{a,\epsilon}(t) = \begin{cases} 0, & t < a \\ \frac{1}{\epsilon}, & a \leq t < a + \epsilon \\ 0, & t \geq a + \epsilon \end{cases}$$

Notice that

$$\int_a^{a+\epsilon} d_{a,\epsilon}(t) dt = \int_a^{a+\epsilon} \frac{1}{\epsilon} dt = 1.$$

$$d_{a,\epsilon}(t) = \frac{1}{\epsilon} [u(t-a) - u(t-(a+\epsilon))]$$

Here's a graph of  $d_{2,1}(t)$ , for example:



Since the unit impulse function is a linear combination of unit step functions, we could solve differential equations with impulse functions so-constructed. As far as Laplace transform goes, it's even easier to take the limit as  $\epsilon \rightarrow 0$  for the Laplace transforms  $\mathcal{L}\{d_{a,\epsilon}(t)\}(s)$ , and this effectively models impulses on very short time scales.

$$\begin{aligned} d_{a,\epsilon}(t) &= \frac{1}{\epsilon} [u(t-a) - u(t-(a+\epsilon))] \\ \Rightarrow \mathcal{L}\{d_{a,\epsilon}(t)\}(s) &= \frac{1}{\epsilon} \left( \frac{e^{-as}}{s} - \frac{e^{-(a+\epsilon)s}}{s} \right) \\ \lim_{\epsilon \rightarrow 0} &= e^{-as} \left( \frac{1 - e^{-\epsilon s}}{\epsilon s} \right). \end{aligned}$$

$$\begin{array}{c|c} f(t-a)u(t-a) & e^{-as} F(s) \\ u(t-a) & e^{-as} \frac{1}{s} \end{array}$$

In Laplace land we can use L'Hopital's rule (in the variable  $\epsilon$ ) to take the limit as  $\epsilon \rightarrow 0$ :

$$\lim_{\epsilon \rightarrow 0} e^{-as} \left( \frac{1 - e^{-\epsilon s}}{\epsilon s} \right) = e^{-as} \lim_{\epsilon \rightarrow 0} \left( \frac{s e^{-\epsilon s}}{s} \right) = e^{-as}.$$

$$\begin{aligned} \frac{d}{d\epsilon} (1 - e^{-\epsilon s}) &= -e^{-\epsilon s} (-s) \\ \frac{d}{d\epsilon} (\epsilon s) &= s \end{aligned}$$

The result in time  $t$  space is not really a function but we call it the "delta function"  $\delta(t-a)$  anyways, and visualize it as a function that is zero everywhere except at  $t=a$ , and that it is infinite at  $t=a$  in such a way that its integral over any open interval containing  $a$  equals one. As explained in EP7.6, the delta "function" can be thought of in a rigorous way as a linear transformation, not as a function. It can also be thought of as the derivative of the unit step function  $u(t-a)$ , and this is consistent with the Laplace table entries for derivatives of functions. In any case, this leads to the very useful Laplace transform table entry

$\delta(t-a)$ unit impulse function	$e^{-as}$	for impulse forcing
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Exercise 1) Revisit the swing from Wednesday's notes and solve the IVP below for  $x(t)$ . In this case the parent is providing an impulse each time the child passes through equilibrium position after completing a cycle.

$$\begin{cases} x''(t) + x(t) = .2\pi [\delta(t) + \delta(t - 2\pi) + \delta(t - 4\pi) + \delta(t - 6\pi) + \delta(t - 8\pi)] \\ x(0) = 0 \\ x'(0) = 0. \end{cases}$$

$$\mathcal{L}: \quad s^2 X(s) + X(s) = .2\pi [1 + e^{-2\pi s} + e^{-4\pi s} + e^{-6\pi s} + e^{-8\pi s}]$$

$$X(s) = .2\pi \left[ \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-4\pi s}}{s^2+1} + \dots + \frac{e^{-8\pi s}}{s^2+1} \right]$$

$$x(t) = .2\pi \left[ \sin t + u(t-2\pi) \sin(t-2\pi) + u(t-4\pi) \sin(t-4\pi) + \dots + u(t-8\pi) \sin(t-8\pi) \right]$$

$$x(t) = .2\pi \sin t \left[ 1 + u(t-2\pi) + u(t-4\pi) + u(t-6\pi) + u(t-8\pi) \right]$$

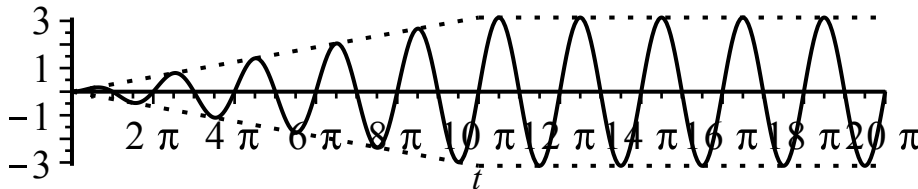
$$t \geq 8\pi$$

$$x(t) = .2\pi \sin t (5) = \pi \sin t$$

f	F
$\frac{k}{s^2+k^2}$	$e^{-as} F(s)$
$u(t-a)f(t-a)$	
$F(s) = \frac{1}{s^2+1}$	
$f(t) = \sin t$	

```
> with(plots):
> plot1 := plot(.1*t*sin(t), t=0..10*Pi, color=black):
> plot2 := plot(Pi*sin(t), t=10*Pi..20*Pi, color=black):
> plot3 := plot(Pi, t=10*Pi..20*Pi, color=black, linestyle=2):
> plot4 := plot(-Pi, t=10*Pi..20*Pi, color=black, linestyle=2):
> plot5 := plot(.1*t, t=0..10*Pi, color=black, linestyle=2):
> plot6 := plot(-.1*t, t=0..10*Pi, color=black, linestyle=2):
> display({plot1, plot2, plot3, plot4, plot5, plot6}, title='Wednesday adventures at the swingset');
```

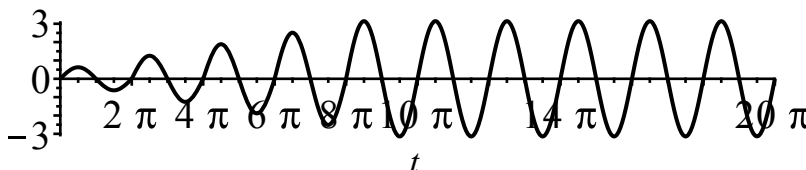
Wednesday adventures at the swingset



impulse solution: five equal impulses to get same final amplitude of  $\pi$  meters - Exercise 1:

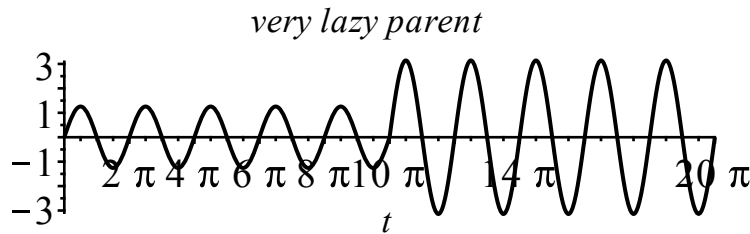
```
> f := t -> .2*Pi*sum(Heaviside(t - k*2*Pi)*sin(t - k*2*Pi), k=0..4):
> plot(f(t), t=0..20*Pi, color=black, title='lazy parent on Friday');
```

lazy parent on Friday



Or, an impulse at  $t = 0$  and another one at  $t = 10\pi$ .

```
> g := t -> .2 * Pi * (2 * sin(t) + 3 * Heaviside(t - 10 * Pi) * sin(t - 10 * Pi)) :
> plot(g(t), t = 0 .. 20 * Pi, color = black, title = 'very lazy parent');
```



### Convolutions and solutions to non-homogeneous physical oscillation problems (EP7.6 p. 499-501)

Consider a mechanical or electrical forced oscillation problem for  $x(t)$ , and the particular solution that begins at rest:

$$\begin{aligned} a x'' + b x' + c x &= f(t) \\ x(0) &= 0 \\ x'(0) &= 0. \end{aligned}$$

Then in Laplace land, this equation is equivalent to

$$\begin{aligned} a s^2 X(s) + b s X(s) + c X(s) &= F(s) \\ \Rightarrow X(s) (a s^2 + b s + c) &= F(s) \\ \Rightarrow X(s) = F(s) \cdot \frac{1}{a s^2 + b s + c} &:= F(s) W(s). \end{aligned}$$

Because of the convolution table entry

$f * g(t) = \int_0^t f(\tau) g(t - \tau) d\tau$	$F(s) G(s)$	convolution integrals to invert Laplace transform products
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$= g * f(t) = \int_0^t g(\tau) f(t - \tau) d\tau$  (you can check with change of variables)  
the solution is given by (has to be true, because  $F(s) G(s) = G(s) F(s)$ )

$$x(t) = f * w(t) = w * f(t) = \int_0^t w(\tau) f(t - \tau) d\tau.$$

where  $w(t) = \mathcal{L}^{-1}\{W(s)\}(t)$ . The function  $w(t)$  is called the "weight function" of the differential equation, because the solution  $x(t)$  is some sort of weighted average of the the forces  $f$  between times 0 and  $t$ , where the weighting factors are given by  $w$  in some sort of convoluted way.

This idea generalizes to much more complicated mechanical and circuit systems, and is how engineers experiment mathematically with how proposed configurations will respond to various input forcing functions, once they figure out the weight function for their system.

The mathematical justification for the general convolution table entry is at the end of today's notes, for those who have studied iterated double integrals and who wish to understand it.

Exercise 2. Let's play the resonance game and practice convolution integrals, first with an old friend, but then with non-sinusoidal forcing functions. We'll stick with our earlier swing, but consider various forcing periodic functions  $f(t)$ .

$$\begin{aligned}x''(t) + x(t) &= f(t) \\ x(0) &= 0 \\ x'(0) &= 0\end{aligned}$$

9.8 m  
 $\omega_0 = \sqrt{\frac{g}{L}} = 1$   
 $x_H = C \cos(t - \alpha)$

- a) Find the weight function  $w(t)$ .  
 b) Write down the solution formula for  $x(t)$  as a convolution integral.  
 c) Work out the special case of  $X(s)$  when  $f(t) = \cos(t)$ , and verify that the convolution formula reproduces the answer we would've gotten from the table entry

$\frac{t}{2k} \sin(kt)$	$\frac{s}{(s^2 + k^2)^2}$
$\sin kt$	$\frac{k}{s^2 + k^2}$
$\int_0^t f(\tau) g(t-\tau) d\tau$	$F(s) G(s)$

a)  $\mathcal{L}$  for IVP:

$$s^2 X(s) + X(s) = F(s)$$

$$X(s)(s^2 + 1) = F(s)$$

$$X(s) = \frac{F(s)}{s^2 + 1} = F(s) W(s)$$

$$w(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right)\{t\} = \sin t = w(t)$$

$$b) x(t) = \int_0^t f(\tau) \sin(t-\tau) d\tau = \int_0^t \sin \tau f(t-\tau) d\tau$$

$$c) x(t) = \int_0^t (\cos \tau) (\sin(t-\tau)) d\tau$$

$$\text{Note: } X(s) = \left(\frac{s}{s^2 + 1}\right) \cdot \frac{1}{s^2 + 1} = \frac{s}{(s^2 + 1)^2}$$

"know":  $x(t) = \frac{t}{2} \sin t$   
 check get same answer with convolution

$$\int_0^t \cos \tau [\cos t \sin(t-\tau) + \sin t \cos(t-\tau)] d\tau$$

$$= \int_0^t \cos \tau [-\cos t \sin \tau + \sin t \cos \tau] d\tau$$

$$= -\cos t \int_0^t \cos \tau \sin \tau d\tau + \sin t \int_0^t \cos^2 \tau d\tau$$

$$= -\cos t \left[ \frac{1}{2} (\sin^2 \tau) \right]_0^t + \sin t \int_0^t \frac{1 + \cos 2\tau}{2} d\tau$$

$$= -\cos t \left( \frac{1}{2} \sin^2 t - 0 \right) + \sin t \left[ \frac{\tau}{2} + \frac{1}{4} \sin 2\tau \right]_0^t$$

$$= -\frac{1}{2} \cos t \sin^2 t + \left[ \frac{t}{2} \sin t + \frac{1}{4} \sin t \sin t \right]$$

$$(1) \quad 2 \sin t \cos t$$

$$\frac{1}{2} \sin(t) t$$

$$\frac{1}{2} \sin(t) t$$

$$\begin{aligned}> \int_0^t \sin(\tau) \cos(t-\tau) d\tau; \\ &\int_0^t \cos(\tau) \sin(t-\tau) d\tau; \# \text{convolution is commutative}\end{aligned}$$

d) Then play the resonance game on the following pages with new periodic forcing functions ...

We worked out that the solution to our DE IVP will be

$$x(t) = \int_0^t \sin(\tau) f(t - \tau) d\tau$$

$$\begin{cases} x'' + x = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{cases} \quad \omega_0 = 1$$

Since the unforced system has a natural angular frequency  $\omega_0 = 1$ , we expect resonance when the forcing function has the corresponding period of  $T_0 = \frac{2\pi}{\omega_0} = 2\pi$ . We will discover that there is the possibility for resonance if the period of  $f$  is a **multiple** of  $T_0$ . (Also, forcing at the natural period doesn't guarantee resonance...it depends what function you force with.)

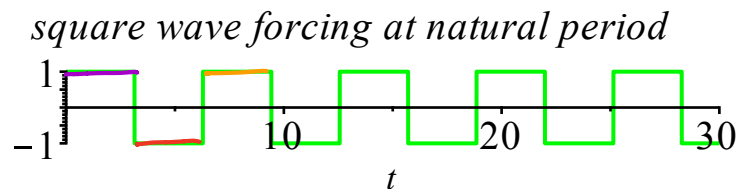
**Example 1)** A square wave forcing function with amplitude 1 and period  $2\pi$ . Let's talk about how we came up with the formula (which works until  $t = 11\pi$ ).

> with (plots) :

>  $f1 := t \rightarrow -1 + 2 \cdot \left( \sum_{n=0}^{10} (-1)^n \cdot \text{Heaviside}(t - n \cdot \text{Pi}) \right) : = -1 + 2 \left( u(t) - u(t - \pi) + u(t - 2\pi) - u(t - 3\pi) + \dots \right)$

$\text{plot1a} := \text{plot}(f1(t), t = 0..30, \text{color} = \text{green}) :$

$\text{display}(\text{plot1a}, \text{title} = \text{'square wave forcing at natural period'}) ;$

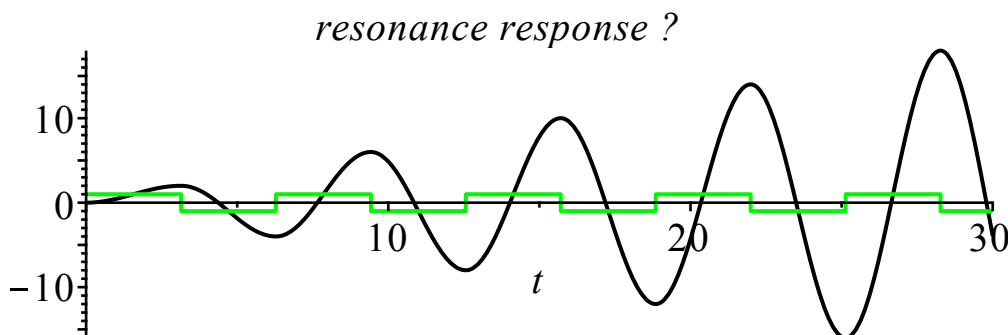


1) What's your vote? Is this square wave going to induce resonance, i.e. a response with linearly growing amplitude?

>  $x1 := t \rightarrow \int_0^t \sin(\tau) \cdot f1(t - \tau) d\tau :$

$\text{plot1b} := \text{plot}(x1(t), t = 0..30, \text{color} = \text{black}) :$

$\text{display}(\{\text{plot1a}, \text{plot1b}\}, \text{title} = \text{'resonance response ?'}) ;$



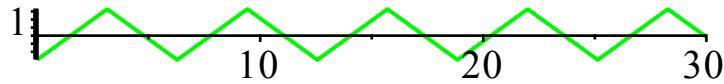
**Example 2)** A triangle wave forcing function, same period

```
> f2 := t → ∫0t f1(s) ds - 1.5 : # this antiderivative of square wave should be triangle wave
```

```
plot2a := plot(f2(t), t = 0..30, color = green) :
```

```
display(plot2a, title = `triangle wave forcing at natural period`);
```

*triangle wave forcing at natural period*



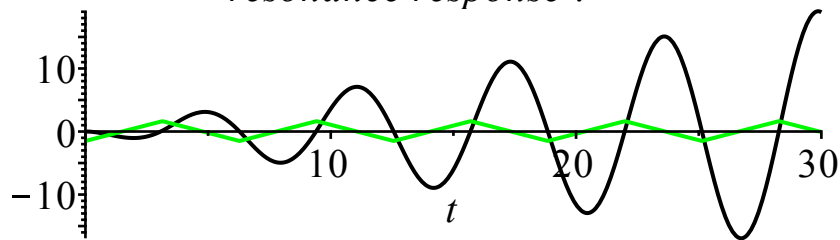
2) Resonance?

```
> x2 := t → ∫0t sin(τ) · f2(t - τ) dτ :
```

```
plot2b := plot(x2(t), t = 0..30, color = black) :
```

```
display( {plot2a, plot2b}, title = `resonance response ?`);
```

*resonance response ?*



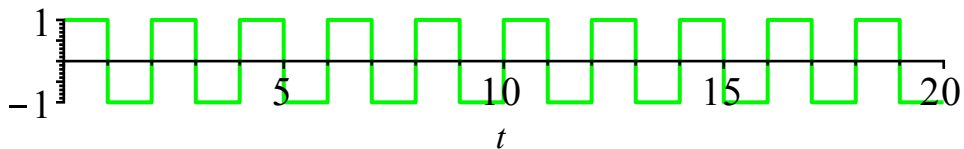
**Example 3)** Forcing not at the natural period, e.g. with a square wave having period  $T = 2$ .

```
> f3 := t -> -1 + 2 * sum_{n=0}^{20} (-1)^n * Heaviside(t - n) :
```

```
plot3a := plot(f3(t), t = 0..20, color = green) :
```

```
display(plot3a, title = `periodic forcing, not at the natural period`);
```

*periodic forcing, not at the natural period*



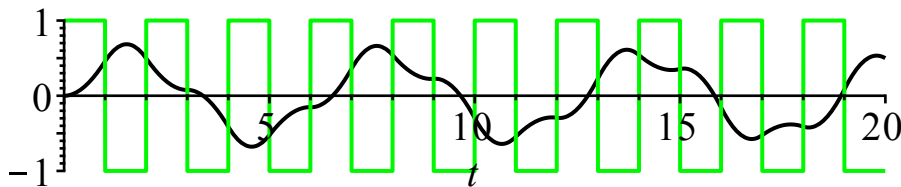
3) Resonance?

```
> x3 := t -> integral_0^t sin(tau) * f3(t - tau) dtau :
```

```
plot3b := plot(x3(t), t = 0..20, color = black) :
```

```
display({plot3a, plot3b}, title = `resonance response ?`);
```

*resonance response ?*





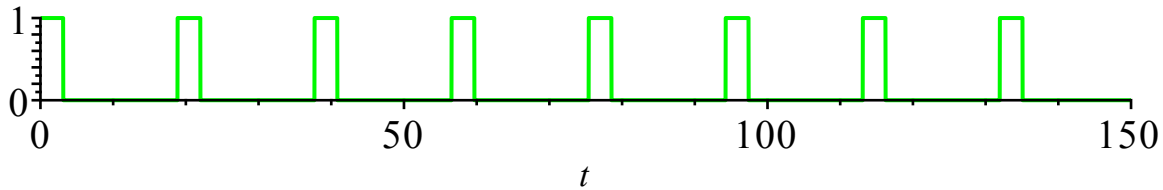
**Example 4)** Forcing not at the natural period, e.g. with a particular wave having period  $T = 6\pi$ .

```
> f4 := t -> sum_{n=0}^{10} (Heaviside(t - 6 * n * pi) - Heaviside(t - (6 * n + 1) * pi)) :
```

```
plot4a := plot(f4(t), t = 0..150, color = green) :
```

```
display(plot4a, title = sporadic square wave with period 6pi);
```

*sporadic square wave with period  $6\pi$*



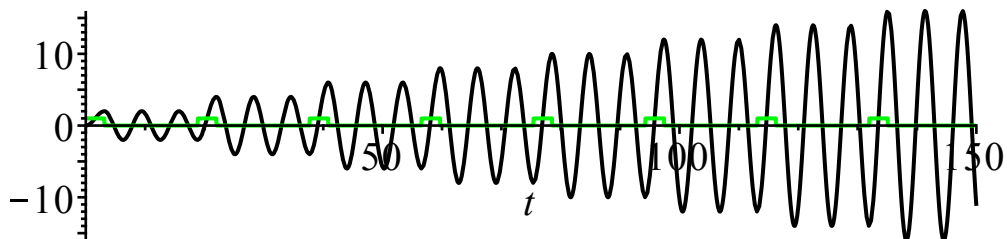
4) Resonance?

```
> x4 := t -> int_0^t sin(tau) * f4(t - tau) dtau :
```

```
plot4b := plot(x4(t), t = 0..150, color = black) :
```

```
display({plot4a, plot4b}, title = `resonance response ?`);
```

*resonance response ?*



**Hey, what happened????** How do we need to modify our thinking if we force a system with something which is not sinusoidal, in terms of worrying about resonance? In the case that this was modeling a swing (pendulum), how is it getting pushed?

**Precise Answer:** It turns out that any periodic function with period  $P$  is a (possibly infinite) superposition of a constant function with *cosine* and *sine* functions of periods  $\left\{P, \frac{P}{2}, \frac{P}{3}, \frac{P}{4}, \dots\right\}$ . Equivalently, these functions in the superposition are

$\{1, \cos(\omega t), \sin(\omega t), \cos(2\omega t), \sin(2\omega t), \cos(3\omega t), \sin(3\omega t), \dots\}$  with  $\omega = \frac{2\pi}{P}$ . This is the theory of Fourier series, which you will study in other courses, e.g. Math 3150, Partial Differential Equations. If the given periodic forcing function  $f(t)$  has non-zero terms in this superposition for which  $n\omega = \omega_0$  (the natural angular frequency) (equivalently  $\frac{P}{n} = \frac{2\pi}{\omega_0} = T_0$ ), there will be resonance;

otherwise, no resonance. We could already have understood some of this in Chapter 5, for example

Exercise 3) The natural period of the following DE is (still)  $T_0 = 2\pi$ . Notice that the period of the first forcing function below is  $T = 6\pi$  and that the period of the second one is  $T = T_0 = 2\pi$ . Yet, it is the first DE whose solutions will exhibit resonance, not the second one. Explain, using Chapter 5 superposition ideas.

a)

$$x''(t) + x(t) = \cos(t) + \sin\left(\frac{t}{3}\right).$$

Resonance  $T = 6\pi$

b)

$$x''(t) + x(t) = \cos(2t) - 3\sin(3t).$$

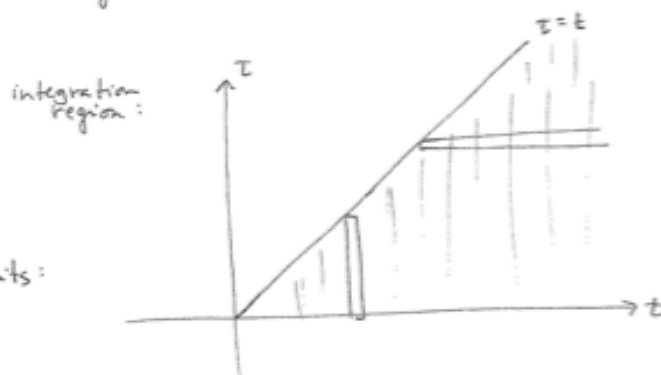
does not

$T = 2\pi$

proof of convolution theorem:

(is a good review of iterated integrals)

$$\begin{aligned} \mathcal{L}\{f * g\}(s) &= \int_0^{\infty} e^{-st} \left( \int_0^t f(\tau) g(t-\tau) d\tau \right) dt \\ &= \int_0^{\infty} \int_0^t e^{-st} f(\tau) g(t-\tau) d\tau dt \end{aligned}$$



interchange limits:

$$\begin{aligned} &= \int_0^{\infty} \int_{\tau}^{\infty} e^{-st} f(\tau) g(t-\tau) dt d\tau \\ &= \int_0^{\infty} \int_{\tau}^{\infty} e^{-s\tau} f(\tau) e^{-s(t-\tau)} g(t-\tau) dt d\tau \quad (\text{pattern recognition}) \\ &= \int_0^{\infty} e^{-s\tau} f(\tau) \left[ \int_{\tau}^{\infty} e^{-s(t-\tau)} g(t-\tau) dt \right] d\tau \\ &\quad \begin{array}{l} \tilde{t} = t - \tau \\ d\tilde{t} = dt \end{array} \\ &\quad \underbrace{\left[ \int_0^{\infty} e^{-s\tilde{t}} g(\tilde{t}) d\tilde{t} \right]}_{G(s)} \\ &= G(s) \int_0^{\infty} e^{-s\tau} f(\tau) d\tau \\ &= G(s) F(s) \quad !! \end{aligned}$$

- postpone Chapter 6 HW 'til next week (start 6 tomorrow)
- today: convolutions

Math 2250-004

Week 13.

Mon Apr 10: We will use last Friday's notes to discuss of finish discussing impulse forcing and the convolution table entry for Laplace transforms. If we have time, we'll move ahead into Tuesday's notes.

Because it is so important it's worth highlighting that for any non-homogeneous forced oscillation problem, the solution function  $x(t)$  for the IVP that starts with zero initial conditions can be written as a convolution integral: For the IVP

$$\text{IVP} \left\{ \begin{array}{l} a x'' + b x' + c x = f(t) \\ x(0) = 0 \\ x'(0) = 0 \end{array} \right. \quad \begin{array}{l} x(0) = x_0 \\ v(0) = v_0 \end{array}$$

$$\begin{array}{l|l} x''(t) & s^2 X(s) - s x(0) - x'(0) \\ x'(t) & s X(s) - x(0) \end{array}$$

Then in Laplace land, this equation is equivalent to

$$\begin{aligned} a s^2 X(s) + b s X(s) + c X(s) &= F(s) \\ \Rightarrow X(s) (a s^2 + b s + c) &= F(s) \\ \Rightarrow X(s) &= F(s) \cdot \frac{1}{a s^2 + b s + c} = F(s) \cdot W(s) \end{aligned}$$

+ stuff with  $x_0$  &  $v_0$   
 $\frac{1}{a s^2 + b s + c}$

for

$$W(s) := \frac{1}{a s^2 + b s + c}.$$

The inverse Laplace transform  $w(t) = \mathcal{L}^{-1}\{W(s)\}(t)$  is called the "weight function" of the given differential equation. Notice (check!) that  $w(t)$  is the solution to the homogeneous DE IVP

$$\begin{aligned} a x'' + b x' + c x &= 0 \\ x(0) &= 0 \\ x'(0) &= 1/a \end{aligned}$$

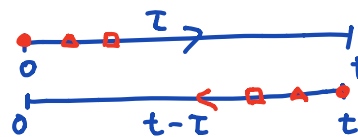
$$\begin{aligned} \mathcal{L}: \quad a(s^2 X(s) - 1/a) + b(s X(s)) + c X(s) &= 0 \\ X(s) [a s^2 + b s + c] &= 1 \\ W(s) = X(s) &= \frac{1}{a s^2 + b s + c} \quad \checkmark \end{aligned}$$

Because of the convolution table entry

$f * g(t) = \int_0^t f(\tau) g(t - \tau) d\tau$	$F(s) G(s)$	convolution integrals to invert Laplace transform products
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the solution (for ANY forcing function  $f(t)$ ) is given by

$$x(t) = \int_0^t f(\tau) w(t - \tau) d\tau.$$



(With non-zero initial conditions there would be homogeneous solution terms as well. In the case of damping these terms would be transient.) Notice that this says that  $x(t)$  depends on the values of the forcing function  $f(\tau)$  for the previous times  $0 \leq \tau \leq t$ , weighted by  $w(t - \tau)$ ,  $t \geq t - \tau \geq 0$ . That the non-homogenous solutions can be constructed from the homogeneous ones via this convolution is a special case of "Duhamel's Principle", which applies to linear DE's and linear PDE's:

[https://en.wikipedia.org/wiki/Duhamel%27s\\_principle](https://en.wikipedia.org/wiki/Duhamel%27s_principle)