Name________________________
Student I.D.___________________

Math 2250-010
Super Quiz 3
April 18, 2014 SOLUTIONS

*1) Solve **one** of the two following Laplace transform problems. There is a Laplace transform table at the end of the superquiz. If you try both problems, indicate clearly which one you wish to have graded.*
(5 points)

1a) Use Laplace transform to solve this forced oscillation initial value problem.

$$x''(t) + 5\,x'(t) + 6\,x(t) = \begin{cases} 2 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$
$$x(0) = 0$$
$$x'(0) = 0$$

Solution: Notice that we can rewrite the piecewise forcing function as $f(t) = 2 - 2\,u(t-1)$. Then, since the solution to this IVP makes both sides of the differential equation equal, their Laplace transforms will be too, i.e.

$$s^2X(s) - 0\,s - 0 + 5(sX(s) - 0) + 6\,X(s) = \frac{2}{s} - \frac{2}{s}\mathrm{e}^{-s}$$
$$X(s)\left(s^2 + 5\,s + 6\right) = \frac{\left(1 - \mathrm{e}^{-s}\right)2}{s}$$
$$X(s) = \left(1 - \mathrm{e}^{-s}\right)\frac{2}{\left(s^2 + 5\,s + 6\right)s}\,.$$

partial fractions:

$$\frac{2}{(s+3)(s+2)s} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s}\,.$$

Equating what the numerators would be if the RHS was written as a fraction using the denominator on the LHS yields

$$2 = A\,s(s+3) + B\,s(s+2) + C(s+2)(s+3)$$

@ $s = 0, -2, -3$ yields

$$2 = 6\,C \Rightarrow C = \frac{1}{3}$$
$$2 = -2\,A \Rightarrow A = -1$$
$$2 = 3\,B \Rightarrow B = \frac{2}{3}\,.$$
$$\Rightarrow X(s) = \left(1 - \mathrm{e}^{-s}\right)\left(-\frac{1}{s+2} + \frac{2}{3}\,\frac{1}{s+3} + \frac{1}{3}\,\frac{1}{s}\right)$$

The inverse Laplace transform of

$$-\frac{1}{s+2} + \frac{2}{3}\,\frac{1}{s+3} + \frac{1}{3}\,\frac{1}{s}$$

is $g(t) = -\mathrm{e}^{-2t} + \frac{2}{3}\mathrm{e}^{-3t} + \frac{1}{3}$. Therefore, using the "translation" table entry, the inverse Laplace transform of

$$e^{-s}\left(-\frac{1}{s+2}+\frac{2}{3}\frac{1}{s+3}+\frac{1}{3}\frac{1}{s}\right)$$

is $u(t-1)g(t-1)$, i.e.

$$u(t-1)\left(-e^{-2(t-1)}+\frac{2}{3}e^{-3(t-1)}+\frac{1}{3}\right).$$

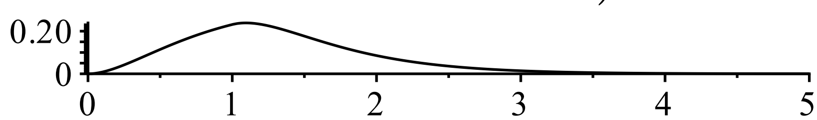
Thus the solution to the IVP is

$$x(t)=-e^{-2t}+\frac{2}{3}e^{-3t}+\frac{1}{3}+u(t-1)\left(-e^{-2(t-1)}+\frac{2}{3}e^{-3(t-1)}+\frac{1}{3}\right).$$

> *with*(*DEtools*) : *#tech check*
> *dsolve*($\{x''(t)+5\cdot x'(t)+6\cdot x(t)=2-2\cdot\text{Heaviside}(t-1), x(0)=0, x'(0)=0\}$);

$$x(t)=-e^{-2t}+\frac{2}{3}e^{-3t}+\frac{1}{3}-\frac{1}{3}\text{Heaviside}(t-1)+\text{Heaviside}(t-1)e^{-2t+2}-\frac{2}{3}\text{Heaviside}(t-1)e^{-3t+3} \quad \textbf{(1)}$$

> *plot*$\left(-e^{-2t}+\frac{2}{3}e^{-3t}+\frac{1}{3}-\frac{1}{3}\text{Heaviside}(t-1)+\text{Heaviside}(t-1)e^{-2t+2}-\frac{2}{3}\text{Heaviside}(t-1)e^{-3t+3}, t=0..5, color=black\right)$; *#solution plot*



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1b) Find an integral convolution formula that will work for any forcing function $f(t)$, for the solution $x(t)$ to

$$x''(t)+5x'(t)+6x(t)=f(t)$$
$$x(0)=0$$
$$x'(0)=0$$

Solution: proceeding as in 1a, one finds that

$$X(s)=\frac{1}{s^2+5s+6}F(s)=\frac{1}{(s+3)(s+2)}F(s)=F(s)G(s).$$

Since

$$G(s)=\frac{1}{(s+3)(s+2)}=\frac{1}{s+2}-\frac{1}{s+3}$$

the inverse Laplace transform

$$g(t)=e^{-2t}-e^{-3t}.$$

Therefore the solution to the IVP is given by

$$x(t)=f*g(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau=\int_0^t f(\tau)\left(e^{-2(t-\tau)}-e^{-3(t-\tau)}\right)d\tau$$

Since convolution is commutative, you could also write $x(t)=g*f(t)$.

2a) Find the eigenvalues and eigenvectors (eigenspace bases) for the following matrix

$$\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}$$

Hint: if you compute your characteristic polynomial correctly you will find that the eigenvalues are negative integers.

Solution: Start by finding the characterstic polynomial and its roots, the eigenvalues:

$$\begin{vmatrix} -3-\lambda & 1 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda+3)(\lambda+4) - 2 = \lambda^2 + 7\lambda + 10 = (\lambda+5)(\lambda+2) .$$

Thus the eigenvalues are $\lambda = -5, -2$

$E_{\lambda=-5}$ is the solution space to the homogeneous system with matrix

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

and a basis for this eigenspace is $\underline{\mathbf{v}} = [-1, 2]^T$.

$E_{\lambda=-2}$ is the solution space to the homogeneous system with matrix

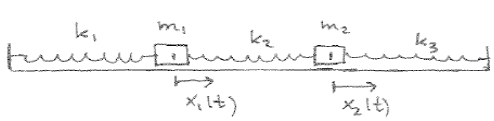
$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

and a basis for this eigenspace is $\underline{\mathbf{v}} = [1, 1]^T$.

(3 points)

2b) Set $m_1 = 1$, $k_1 = 2$. Find values for the other mass m_2 and for the other two spring constants k_2, k_3, so that the displacements $x_1(t)$, $x_2(t)$ of the two masses from equilibrium in the configuration below satisfy

$$x_1''(t) = -3x_1 + x_2$$
$$x_2''(t) = 2x_1 - 4x_2$$



solution: From Newton's laws, the system of DE's for $x_1(t), x_2(t)$ are

$$m_1 x_1''(t) = -k_1 x_1 + k_2(x_2 - x_1) = -(k_1 + k_2)x_1 + k_2 x_2$$
$$m_2 x_2''(t) = -k_2(x_2 - x_1) - k_3 x_2 = k_2 x_1 - (k_2 + k_3)x_2$$

i.e.

$$x_1''(t) = -\frac{(k_1 + k_2)}{m_1}x_1 + \frac{k_2}{m_1}x_2$$

$$x_2''(t) = \frac{k_2}{m_2}x_1 - \frac{(k_2 + k_3)}{m_2}x_2 .$$

Since $m_1 = 1$ and $k_1 = 2$ the first equation simplifies to

$$x_1''(t) = -(2+k_2)x_1 + k_2 x_2$$

which is supposed to be

$$x_1''(t) = -3\,x_1 + x_2$$

so $k_2 = 1$. Thus the second DE simplifies to

$$x_2''(t) = \frac{1}{m_2}x_1 - \frac{(1+k_3)}{m_2}x_2$$

which is supposed to be

$$x_2''(t) = 2\,x_1 - 4\,x_2$$

so $m_2 = \frac{1}{2}$, $k_3 = 1$.

(2 points)

2c) Find the general solution to

$$x_1''(t) = -3\,x_1 + x_2$$
$$x_2''(t) = 2\,x_1 - 4\,x_2$$

(3 points)

solution: In matrix-vector form this is the mass-spring system

$$\begin{bmatrix} x_1''(t) \\ x_2''(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and we know the eigenvalues and eigenvectors from 2a)

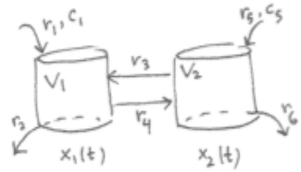
$$E_{\lambda=-5} = span\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\},\quad E_{\lambda=-2} = span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

For this second order DE describing a conservative mass-spring system, we get solutions $\cos(\omega t)\underline{\mathbf{v}}$, $\sin(\omega t)\underline{\mathbf{v}}$ where v is an eigenvector of the acceleration matrix A, with eigenvalue λ and $\omega = \sqrt{-\lambda}$. Thus the general solution is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \left(c_1\cos\sqrt{2}\,t + c_2\sin\sqrt{2}\,t\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(c_3\cos\sqrt{5}\,t + c_4\sin\sqrt{5}\,t\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(Notice the slow mode has both masses in phase and with equal amplitude, and angular frequency $\omega = \sqrt{2}$; the fast mode has both masses out of phase, with the second mass oscillating at twice the amplitude of the first mass, and angular frequency $\omega = \sqrt{5}$.)

3) Consider a general input-output model with two compartments as indicated below. The compartments contain volumes V_1, V_2 and solute amounts $x_1(t)$, $x_2(t)$ respectively. The flow rates (volume per time) are indicated by r_i, $i = 1..6$. The two input concentrations (solute amount per volume) are c_1, c_5.



3a) *Suppose* $r_2 = r_3 = 100,\ r_1 = r_4 = r_5 = 200,\ r_6 = 300\ \dfrac{m^3}{hour}$. *Explain why the volumes* $V_1(t),\ V_2(t)$ *remain constant.*

(2 points)

We have

$$V_1'(t) = r_1 + r_3 - r_2 - r_4 = 200 + 100 - 100 - 200 = 0$$
$$V_2'(t) = r_4 + r_5 - r_3 - r_6 = 200 + 200 - 100 - 300 = 0\ .$$

Since $V_1'(t), V_2'(t)$ are identically zero, V_1, V_2 are constant.

3b) *Using the flow rates above, incoming concentrations* $c_1 = 0.05,\ c_5 = 0\ \dfrac{kg}{m^3}$, *volumes* $V_1 = V_2 = 100\ m^3$, *show that the amounts of solute* $x_1(t)$ *in tank 1 and* $x_2(t)$ *in tank 2 satisfy*

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

(4 points)

$$x_1'(t) = r_1 c_1 + r_3 \frac{x_2}{V_2} - (r_2 + r_4)\frac{x_1}{V_1} = 200 \cdot .05 + 100\frac{x_2}{100} - (100 + 200)\frac{x_1}{100} = 10 - 3x_1 + x_2$$
$$x_2'(t) = r_5 c_5 + r_4 \frac{x_1}{V_1} - (r_3 + r_6)\frac{x_2}{V_2} = 200 \cdot 0 + 200\frac{x_1}{100} - (100 + 300)\frac{x_2}{100} = 2x_1 - 4x_2\ .$$

The two differential equations are equivalent to the displayed system that is written in vector-matrix form.

3c) *Verify that a particular solution to this system of differential equations is given by the constant vector function*

$$\underline{\boldsymbol{x}}_P = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

(2 points)

solution: (One should be able to find such a particular solution as well, but here it was given to us so we only need to check that this constant vector function makes the differential equation true. For

$$\underline{\boldsymbol{x}}_P(t) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

We have $\underline{\boldsymbol{x}}_P'(t) = \underline{\boldsymbol{0}}$ (the left side of the differential equation). On the other hand the right side of the differential equation in this case is

$$\begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \end{bmatrix} = \begin{bmatrix} -12+2+10 \\ 8-8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, since the differential equation is true for the constant vector function $\underline{\mathbf{x}}_P(t)$, this constant is a particular solution.

3d) Find the general solution to the system of differential equations in 3b. Hints: Use the particular solution from 3c as part of your solution, and notice that the matrix in this system is the same as the one in problem 2, so you can use the eigendata you found in that problem

(4 points)

solution: By the fundamental theorem for linear operators, the general solution is

$$x(t) = x_P(t) + x_H(t) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + c_1 e^{-5t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2\, e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 10.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt - kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt + kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\dfrac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\!\left[\dfrac{t}{a}\right]\!\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		