Name________________________
Student I.D.___________________

Math 2250-010
Super Quiz 2 SOLUTIONS
March 7, 2014

1a) What two properties must hold for a subset W of a vector space V in order that W be a subspace?

(1 point)

W must be closed under addition and scalar multiplication.

In other words, it must be true that
α) $\underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in W \Rightarrow \underline{\boldsymbol{u}} + \underline{\boldsymbol{v}} \in W$
β) $\underline{\boldsymbol{u}} \in W, c \in \mathbb{R} \Rightarrow c\underline{\boldsymbol{u}} \in W$

1b) What does it mean for a collection vectors $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \ldots \underline{\boldsymbol{v}}_n$ to be a basis for a vector space/subspace W?

(1 point)

They must span W and be linearly independent.

In other words, each $\underline{\boldsymbol{w}} \in W$ *must be expressable as a linear combination*

$$\underline{\boldsymbol{w}} = c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots c_n\underline{\boldsymbol{v}}_n$$

and it must be true that

$$c_1\underline{\boldsymbol{v}}_1 + c_2\underline{\boldsymbol{v}}_2 + \ldots c_n\underline{\boldsymbol{v}}_n = \underline{\boldsymbol{0}} \Rightarrow c_1 = c_2 = \ldots = c_n = 0.$$

2a) The span of the three vectors

$$\underline{\boldsymbol{v}}_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \underline{\boldsymbol{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \underline{\boldsymbol{v}}_3 = \begin{bmatrix} 4 \\ 1 \\ 12 \\ 1 \end{bmatrix}$$

is a subspace of $\mathbb{R}^4$. Find a basis for this subspace using some or all of the three vectors, and explain your reasoning. Hint: the following matrix, which has the three vectors as its columns, along with its reduced row echelon form on the right, can help you answer this question:

$$\begin{bmatrix} 2 & 0 & 4 \\ -1 & 1 & 1 \\ 3 & 2 & 12 \\ 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(5 points)

Because column dependencies are preserved by elementary row operations, we use the reduced row echelon form to deduce that

$$\underline{\boldsymbol{v}}_3 = 2\,\underline{\boldsymbol{v}}_1 + 3\,\underline{\boldsymbol{v}}_2$$

Thus span $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3\}$ = *span* $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2\}$. Since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2$ are linearly independent it follows that they are a basis of the subspace

$$V = span\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, \underline{\boldsymbol{v}}_3\}.$$

(In fact, any two of the three vectors would have been a basis for this subspace, since each of the three vectors can be written as a linear combination of the other two, and any two of them are linearly independent.)

2b) What is the dimension of the subspace in part 2a?

(1 point)

Since we found a basis consisting of two vectors, the dimension is 2.

3) Consider the following differential equation for a function $x(t)$, which could arise from an unforced mass-spring configuration.

$$x''(t) + 2x'(t) + 5x(t) = 0.$$

3a) Find the general solution to this homogeneous linear differential equation.

(6 points)

The characteristic polynomial is $p(r) = r^2 + 2r + 5 = (r+1)^2 + 4$. The roots are given by

$$\begin{aligned} (r+1)^2 + 4 &= 0 \\ (r+1)^2 &= -4 \\ r+1 &= \pm 2i \\ r &= -1 \pm 2i. \end{aligned}$$

Thus we get a basis of solutions $x_1(t) = e^{-t}\cos(2t), x_2(t) = e^{-t}\sin(2t)$ *and the general (homogenous) solutions consists of all linear combinations*

$$x(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t).$$

(If you're still confused where the "real" solutions came from, recall that from the roots of the characteristic polynomial we get complex exponential solutions

$$\begin{aligned} e^{(-1+2i)t} &= e^{-t}(\cos(2t) + i\sin(2t)) \\ e^{(-1-2i)t} &= e^{-t}(\cos(2t) - i\sin(2t)). \end{aligned}$$

Adding these two solutions and dividing by 2 gives us the "real" solution $e^{-t}\cos(2t)$*; Subtracting these two solutions and dividing by* $2i$ *gives the "real" solution* $e^{-t}\sin(2t)$. *Since this is a homogeneous linear DE, linear combinations of solutions are solutions.)*

3b) Which of the three damping phenomena is exhibited by solutions to this differential equation?

(1 point)

Since the roots of the characteristic polynomial are complex, these solutions are <u>*underdamped*</u>*.....their "amplitude" decays exponentially, but they oscillate infinitely often.*

3c) Now consider the inhomogeneous DE

$$x''(t) + 2x'(t) + 5x(t) = 20.$$

Notice that $x_p(t) = 4$ is a particular solution. Use this fact and your previous work to write down the general solution to the inhomogeneous DE.

(2 points)

(You weren't asked to check, but if substitute $x_P(t) = 4$ into the left side of the DE you get $0 + 0 + 5\cdot 4 = 20$, so this indeed a particular solution.)

$$x(t) = x_P(t) + x_H(t) = 4 + c_1 e^{-t}\cos(2\,t) + c_2 e^{-t}\sin(2\,t) \quad .$$

3d) Solve the initial value problem

$$x''(t) + 2\,x'(t) + 5\,x(t) = 20$$
$$x(0) = 0$$
$$x'(0) = 0.$$

(5 points)

$$x(t) = 4 + c_1 e^{-t}\cos(2\,t) + c_2 e^{-t}\sin(2\,t)$$
$$x'(t) = c_1\left(-e^{-t}\cos(2\,t) - 2e^{-t}\sin(2\,t)\right) + c_2\left(-e^{-t}\sin(2\,t) + 2e^{-t}\cos(2\,t)\right)$$
$$x(0) = 0 = 4 + c_1 \Rightarrow c_1 = -4$$
$$x'(0) = 0 = -c_1 + 2\,c_2 = 4 + 2\,c_2 \Rightarrow 2\,c_2 = -4 \Rightarrow c_2 = -2\,.$$
$$x(t) = 4 - 4\,e^{-t}\cos(2\,t) - 2\,e^{-t}\sin(2\,t) \quad .$$

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> with(DEtools) :
  dsolve({x''(t) + 2·x'(t) + 5·x(t) = 20, x(0) = 0, x'(0) = 0});
     #quick check, although solution looks correct...
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$$x(t) = -2\,e^{-t}\sin(2\,t) - 4\,e^{-t}\cos(2\,t) + 4 \qquad (1)$$

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>
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4) Find the amplitude-phase form $x(t) = C\cos(3\,t - \alpha)$, for the function

$$x(t) = -\cos(3\,t) + \sqrt{3}\sin(3\,t).$$

(3 points)

$$A = -1,\, B = \sqrt{3} \Rightarrow C = \sqrt{A^2 + B^2} = 2\,.$$
$$\frac{A}{C} = \cos(\alpha) = -\frac{1}{2},\ \frac{B}{C} = \sin(\alpha) = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{2\,\pi}{3}\,.$$

If you didn't recognize α as an elementary angle, the following symbolic answers are also correct, for (A, B) in the second quadrant:

$$\alpha = \arccos\left(\frac{A}{C}\right) = \arccos\left(-\frac{1}{2}\right)$$
$$\alpha = \arctan\left(\frac{B}{A}\right) + \pi = \arctan\left(-\sqrt{3}\right) + \pi\,.$$

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> evalf(2·π/3.);
  arctan(-√3.) + evalf(π);
  arccos(-.5);
```

2.094395103

2.094395103

2.094395102 (2)

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>
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