Math 2250-010
Super Quiz 1 SOLUTiONS
January 31, 2014

1a) Solve this initial value problem for a function $y(x)$:

$$y'(x) = \frac{-x}{y+1}$$
$$y(0) = 2$$

(6 points)

DE is separable:

$$(y+1)dy = -x\,dx$$
$$\int (y+1)\,\mathrm{d}y = \int -x\,\mathrm{d}x$$
$$\frac{1}{2}(y+1)^2 = -\frac{x^2}{2} + C$$

(You may have chosen a different antiderivative for the left side - the result of that would be to modify the additive constant on the right.) Multiply both sides of the equation by 2, to clear fractions:

$$(y+1)^2 = -x^2 + C$$

(new C is twice the old one).
Using the initial conditions: $y(0) = 2$:

$$9 = 0 + C \Rightarrow C = 9 \Rightarrow (y+1)^2 = -x^2 + 9.$$

We can rearrange this equation into the standard implicit equation for a circle of radius 3, centered at $(0,-1)$:

$$x^2 + (y+1)^2 = 9.$$

However, we are supposed to find the explicit solution to the initial value problem, so we solve for y:

$$(y+1)^2 = -x^2 + 9.$$
$$y + 1 = \sqrt{9 - x^2}$$

(because $y + 1 = -\sqrt{9 - x^2}$ won't solve $y(0) = 2$.)

$$\Rightarrow y = -1 + \sqrt{9 - x^2}$$

1b) Show that the initial value problem above arises from the following description of the function $y(x)$: The graph of the function $y(x)$ has the property that it goes through the point $(0, 2)$. Furthermore, every normal line to the graph (i.e. every line perpendicular to the graph) passes through the point $(0,-1)$. Hint: perpendicular lines have slopes m_1, m_2 hat are negative reciprocals, $m_1 m_2 = -1$. Thus you can relate the slopes of these normal lines through points (x, y) on the graph to the slopes of the graph at those points.

(2 points)

Since the graph $y = y(x)$ goes through $(0, 2)$ we must have $y(0) = 2$, which is the initial value. For the DE, we can equate the two ways of computing the slope of the normal line to the graph at the point (x, y) on the graph. The first way, using the negative reciprocal property and the fact that the graph has slope $y'(x)$ there, is that the slope of the normal line is given by

$$m_{\perp} = -\frac{1}{y'(x)}.$$

The other way to compute the normal line slope is using the fact that $(0,-1)$ and (x, y) are on this line, so its slope is given by

$$m_{\perp} = \frac{rise}{run} = \frac{y+1}{x}.$$

Thus

$$-\frac{1}{y'(x)} = \frac{y+1}{x} \Rightarrow y'(x) = -\frac{x}{y+1}.$$

2) Consider the following linear drag initial value problem for a velocity function $v(t)$:

$$v'(t) = 30 - 2v$$
$$v(0) = 5.$$

a) Construct and use a phase diagram to determine $\lim_{t\to\infty} v(t)$ for the solution to this IVP.

(2 points)

$$v'(t) = -2(v - 15)$$

so the equilibrium (constant) velocity solution is $v = 15$. For $v > 15$, $v'(t) < 0$; for $v < 15$, $v'(t) > 0$. So the phase diagram is

$$\rightarrow\rightarrow\rightarrow 15 \leftarrow\leftarrow\leftarrow .$$

So, no matter the initial velocity, $\lim_{t\to\infty} v(t) = 15$ (the "terminal velocity" for this linear drag problem).

b) Solve the initial value problem for $v(t)$, using the method for linear differential equations.

(6 points)

$$v'(t) + 2v = 30$$
$$e^{2t}(v'(t) + 2v) = 30 e^{2t}$$
$$\frac{d}{dt}\left(e^{2t} v(t)\right) = 30 e^{2t}$$
$$e^{2t} v = \int 30 e^{2t} dt = 15 e^{2t} + C$$
$$v(t) = 15 + C e^{-2t}$$

$v(0) = 5 \Rightarrow 5 = 15 + C \Rightarrow C = -10 \Rightarrow$

$$v(t) = 15 - 10 e^{-2t}.$$

c) If the position function $x(t)$ corresponding to this velocity function satisfies $x(0) = 0$, find $x(t)$.

$$x(t) - x(0) = \int_0^t x'(s)\, ds = \int_0^t 15 - 10 e^{-2s}\, ds$$

Since $x(0) = 0$,

$$x(t) = \int_0^t 15 - 10 e^{-2s}\, ds = 15s + 5 e^{-2s} \Big]_0^t = 15t + 5\left(e^{-2t} - 1\right).$$

(4 points)

3) Consider a brine tank which initially contains 100 gallons of water, with concentration 0.1 pounds of salt per gallon. At time t = 0 hours water begins to flow into the tank at a rate of 50 gallons per hour, with concentration 0.8 pounds of salt per gallon. At the same time, the well-mixed brine solution begins to flow out of the tank at a rate of 60 gallons per hour, until the tank empties. Let $x(t)$ *be the amount of salt in the tank at time t, until the tank becomes empty.*

a) At what time will the tank become empty?

(1 points)

$$V'(t) = r_i - r_o = 50 - 60 = -10 \ \frac{gal}{hour}$$
$$V(0) = 100$$

Thus $V(t) = \int V'(t)dt = -10\,t + C = -10\,t + 100$ since $V(0) = 100$. Thus $V(t) = 0$ after 10 hours.

b) Find the initial value problem satisfied by the salt amount $x(t)$*. You do not need to find the solution function* $x(t)$ *(because there is not so much time available on this superquiz).*

$$x'(t) = r_i c_i - r_o c_o$$
$$x'(t) = 50 \cdot (0.8) - 60\frac{x(t)}{V(t)}$$
$$x'(t) = 40 - 60\frac{x(t)}{100 - 10\,t}.$$

The initial concentration is 0.1 pound of salt per gallon, and there are initially 100 gallons, so $x(0) = 0.1 \cdot 100 = 10$ pounds of salt.

(4 points)