Math 2250-10
Quiz 3 SOLUTIONS
January 24, 2014

1) Consider the following differential equation for a function $x(t)$. It is not based on the logistic population model, but does have applications that we will discuss very soon.

$$x'(t) = x^2 - x - 2 .$$

Find the equilibrium solutions. Then draw the phase diagram and indicate the stability of the equilibrium solutions.

(3 points)

Solution: Equilibrium (constant) solutions will have zero derivative, so the right side of the differential equation must be zero, i.e.

$$0 = x^2 - x - 2 = (x - 2)(x + 1).$$

Thus the equilibrium solutions are $x \equiv 2, x \equiv -1$. To draw the phase diagram we need to figure out whether $x(t)$ is increasing or decreasing on the intervals between the equilibrium points.
For $x > 2, x'(t) = (x - 2)(x + 1) = (+)(+) = (+)$, so $x(t)$ is increasing.
For $-1 < x < 2, x'(t) = (-)(+) = (-)$, so $x(t)$ is decreasing.
For $x < -1, x'(t) = (-)(-) = (+)$, so $x(t)$ is increasing. Thus the phase diagram is given by

$$\rightarrow \rightarrow -1 \leftarrow \leftarrow 2 \rightarrow \rightarrow.$$

Thus $x \equiv -1$ is asymptotically stable, and $x \equiv 2$ is unstable.

2) Compute the partial fractions decomposition for

$$\frac{1}{(x-2)(x+1)}.$$

You may use either of the methods we've discussed.

(3 points)

Solution: quick way: The partial fraction decomposition will be a multiple of

$$\frac{1}{x-2} - \frac{1}{x+1}$$

since the x-terms will cancel when I recombine:

$$\frac{1}{x-2} - \frac{1}{x+1} = \frac{(x+1)-(x-2)}{(x-2)(x+1)} = \frac{3}{(x-2)(x+1)}.$$

Thus (dividing both sides of the equation above by 3,

$$\frac{1}{(x-2)(x+1)} = \frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right).$$

standard way:

$$\frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1)+B(x-2)}{(x-2)(x+1)}.$$

Thus (equating numerators of the fractions that have the same denominator)

$$1 = A(x+1) + B(x-2).$$

Letting $x = -1 \Rightarrow 1 = -3B \Rightarrow B = -\frac{1}{3}$. Letting $x = 2 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$. Thus

$$\frac{1}{(x-2)(x+1)} = \frac{1}{3}\left(\frac{1}{x-2}\right) - \frac{1}{3}\left(\frac{1}{x+1}\right) = \frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right).$$

3) Use your work from 2 to solve the initial value problem

$$x'(t) = x^2 - x - 2$$
$$x(0) = 1.$$

Using the result of 2, and separating:

$$\frac{dx}{x^2 - x - 2} = dt$$

$$\frac{dx}{(x-2)(x+1)} = dt$$

$$\frac{1}{3}\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx = dt$$

$$\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx = 3\,dt$$

$$\int\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx = \int 3\,dt$$

$$\ln\left|\frac{x-2}{x+1}\right| = 3t + C_1$$

$$\left|\frac{x-2}{x+1}\right| = e^{3t + C_1} = C\,e^{3t}$$

$$\frac{x-2}{x+1} = C\,e^{3t}$$

$x(0) = 1 \Rightarrow \frac{-1}{2} = C \Rightarrow$

$$\frac{x-2}{x+1} = -\frac{1}{2}e^{3t}$$

$$x - 2 = (x+1)\left(-\frac{1}{2}e^{3t}\right) = -\frac{1}{2}x\,e^{3t} - \frac{1}{2}e^{3t}$$

$$x + \frac{1}{2}x\,e^{3t} = 2 - \frac{1}{2}e^{3t}$$

$$x\left(1 + \frac{1}{2}e^{3t}\right) = 2 - \frac{1}{2}e^{3t}$$

$$x(t) = \frac{2 - \frac{1}{2}e^{3t}}{1 + \frac{1}{2}e^{3t}} = \frac{4 - e^{3t}}{2 + e^{3t}} = \frac{4\,e^{-3t} - 1}{2\,e^{-3t} + 1}.$$

(Check: $x(0) = \frac{3}{3} = 1$, as desired. Also, $\lim_{t \to \infty} x(t) = -\frac{1}{1} = -1$, which is consistent with the phase diagram.)

(4 points)