

We will still do more Laplace transform applications - in fact that will be the focus of the labs this week and part of Wednesday's lecture. But the homework for this week will mostly be from Chapter 6, so we will get a good start on that material today.

### 6.1-6.2 Eigenvalues and eigenvectors for square matrices.

The study of eigenvalues and eigenvectors is a return to matrix linear algebra, and the concepts we discuss will help us study linear systems of differential equations, in Chapter 7. Such systems of DE's arise naturally in the contexts of

- coupled input-output models, with several components.
- coupled mass-spring or RLC circuit loops, with several components.

To introduce the idea of eigenvalues and eigenvectors we'll first think geometrically.

Example Consider the matrix transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  with formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Notice that for the standard basis vectors  $\underline{e}_1 = [1, 0]^T$ ,  $\underline{e}_2 = [0, 1]^T$

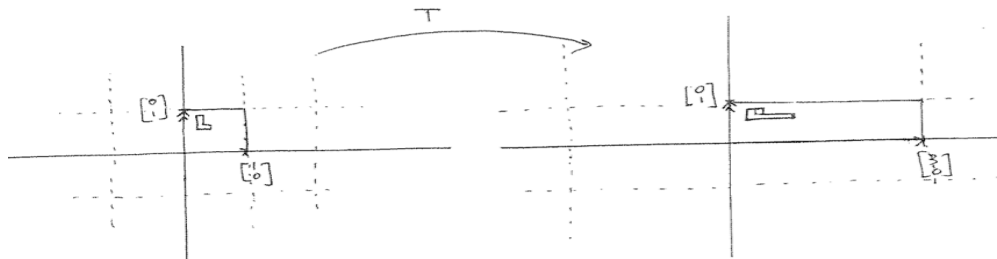
$$T(\underline{e}_1) = 3\underline{e}_1$$

$$T(\underline{e}_2) = \underline{e}_2.$$

The facts that  $T$  is linear and that it transforms  $\underline{e}_1$ ,  $\underline{e}_2$  by scalar multiplying them, lets us understand the geometry of this transformation completely:

$$\begin{aligned} T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T(x_1\underline{e}_1 + x_2\underline{e}_2) = x_1T(\underline{e}_1) + x_2T(\underline{e}_2) \\ &= x_1(3\underline{e}_1) + x_2(1\underline{e}_2). \end{aligned}$$

In other words,  $T$  stretches by a factor of 3 in the  $\underline{e}_1$  direction, and by a factor of 1 in the  $\underline{e}_2$  direction, transforming a square grid in the domain into a parallel rectangular grid in the image:



Exercise 1) Do a similar geometric analysis and sketch for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 2) And for the transformation

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Definition: If  $A_{n \times n}$  and if  $A \underline{v} = \lambda \underline{v}$  for a scalar  $\lambda$  and a vector  $\underline{v} \neq \underline{0}$  then  $\underline{v}$  is called an eigenvector of  $A$ , and  $\lambda$  is called the eigenvalue of  $\underline{v}$ . (In some texts the words characteristic vector and characteristic value are used as synonyms for these words.)

- In the three examples above, the standard basis vectors (or multiples of them) were eigenvectors, and the corresponding eigenvalues were the diagonal matrix entries. A non-diagonal matrix may still have eigenvectors and eigenvalues, and this geometric information can still be important to find. But how do you find eigenvectors and eigenvalues for non-diagonal matrices? ...

Exercise 3) Try to find eigenvectors and eigenvalues for the non-diagonal matrix, by just trying random input vectors  $\underline{x}$  and computing  $A \underline{x}$ .

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

How to find eigenvalues and eigenvectors (including eigenspaces) systematically:

If

$$\begin{aligned} A \underline{v} &= \lambda \underline{v} \\ \Leftrightarrow A \underline{v} - \lambda \underline{v} &= \underline{0} \\ \Leftrightarrow A \underline{v} - \lambda I \underline{v} &= \underline{0} \end{aligned}$$

where  $I$  is the identity matrix.

$$\Leftrightarrow (A - \lambda I) \underline{v} = \underline{0}.$$

As we know, this last equation can have non-zero solutions  $\underline{v}$  if and only if the matrix  $(A - \lambda I)$  is not invertible, i.e.

$$\Leftrightarrow \det(A - \lambda I) = 0.$$

So, to find the eigenvalues and eigenvectors of matrix you can proceed as follows:

- Compute the polynomial in  $\lambda$

$$p(\lambda) = \det(A - \lambda I).$$

If  $A_{n \times n}$  then  $p(\lambda)$  will be degree  $n$ . This polynomial is called the characteristic polynomial of the matrix  $A$ .

- $\lambda_j$  can be an eigenvalue for some non-zero eigenvector  $\underline{v}$  if and only if it's a root of the characteristic polynomial, i.e.  $p(\lambda_j) = 0$ . For each such root, the homogeneous solution space of vectors  $\underline{v}$  solving

$$(A - \lambda_j I) \underline{v} = \underline{0}$$

will be eigenvectors with eigenvalue  $\lambda_j$ . This subspace of eigenvectors will be at least one dimensional, since  $(A - \lambda_j I)$  does not reduce to the identity and so the explicit homogeneous solutions will have free parameters. Find a basis of eigenvectors for this subspace. Follow this procedure for each eigenvalue, i.e. for each root of the characteristic polynomial.

Notation: The subspace of eigenvectors for eigenvalue  $\lambda_j$  is called the  $\lambda_j$  eigenspace, and denoted by  $E_{\lambda_j}$ .

(We include the zero vector in  $E_{\lambda_j}$ .) The basis of eigenvectors is called an eigenbasis for  $E_{\lambda_j}$ .

Exercise 4) a) Use the systematic algorithm to find the eigenvalues and eigenbases for the non-diagonal matrix of Exercise 3.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}.$$

b) Use your work to describe the geometry of the linear transformation in terms of directions that get stretched:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Exercise 5) Find the eigenvalues and eigenspace bases for

$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

- (i) Find the characteristic polynomial and factor it to find the eigenvalues.
- (ii) for each eigenvalue, find bases for the corresponding eigenspaces.
- (iii) Can you describe the transformation  $T(\underline{x}) = B\underline{x}$  geometrically using the eigenbases? Does  $\det(B)$  have anything to do with the geometry of this transformation?

Your solution will be related to the output below:

```
> with(LinearAlgebra) :  
  B := Matrix(3, 3, [4, -2, 1, 2, 0, 1, 2, -2, 3]);  
  Eigenvectors(B);
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$$B := \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

**(1)**

Exercise 6) If your matrix  $A$  is actually diagonal, the general algorithm for finding eigenspace bases just reproduces the entries along the diagonal as eigenvalues, and the corresponding standard basis vectors as eigenspace bases. (Recall our diagonal matrix examples from the start of today's notes, where the standard basis vectors were eigenvectors. This is typical for diagonal matrices.) Illustrate how this works for a  $3 \times 3$  diagonal matrix, so that in the future you can just read off the eigendata if the matrix you're given is (already) diagonal:

$$A := \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}.$$

step 1) Find the roots of the characteristic polynomial  $\det(A - \lambda I)$ .

step 2) Find the eigenspace bases, assuming the values of  $a_{11}, a_{22}, a_{33}$  are distinct (all different). What if  $a_{11} = a_{22}$  but these values do not equal  $a_{33}$ ?

In all of our examples so far, it turns out that by collecting bases from each eigenspace for the matrix  $A_{n \times n}$ , and putting them together, we get a basis for  $\mathbb{R}^n$ . This lets us understand the geometry of the transformation

$$T(\underline{x}) = A \underline{x}$$

almost as well as if  $A$  is a diagonal matrix, and so we call such matrices diagonalizable. Having such a basis of eigenvectors for a given matrix is also extremely useful for algebraic computations, and will give another reason for the word diagonalizable to describe such matrices.

Use the  $\mathbb{R}^3$  basis made of out eigenvectors of the matrix  $B$  in Exercise 5, and put them into the columns of a matrix we will call  $P$ . We could order the eigenvectors however we want, but we'll put the  $E_{\lambda=2}$  basis vectors in the first two columns, and the  $E_{\lambda=3}$  basis vector in the third column:

$$P := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

Now do algebra (check these steps and discuss what's going on!)

$$\begin{aligned} \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & 3 \\ 4 & -4 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \end{aligned}$$

In other words,

$$B P = P D,$$

where  $D$  is the diagonal matrix of eigenvalues (for the corresponding columns of eigenvectors in  $P$ ).

Equivalently (multiply on the right by  $P^{-1}$  or on the left by  $P^{-1}$ ):

$$B = P D P^{-1} \text{ and } P^{-1} B P = D.$$

Exercise 7) Use one of the the identities above to show how  $B^{100}$  can be computed with only two matrix multiplications!

Definition: Let  $A_{n \times n}$ . If there is an  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ) basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  consisting of eigenvectors of  $A$ , then  $A$  is called diagonalizable. This is precisely why we use the word "diagonalizable":

Write  $A \mathbf{v}_j = \lambda_j \mathbf{v}_j$  (some of these  $\lambda_j$  may be the same, as in the previous example). Let  $P$  be the matrix

$$P = [\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n].$$

Then, using the various ways of understanding matrix multiplication, we see

$$\begin{aligned} AP &= A[\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n] = [\lambda_1 \mathbf{v}_1 | \lambda_2 \mathbf{v}_2 | \dots | \lambda_n \mathbf{v}_n] \\ &= [\mathbf{v}_1 | \mathbf{v}_2 | \dots | \mathbf{v}_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}. \\ AP &= PD \\ A &= PD P^{-1} \\ P^{-1}AP &= D. \end{aligned}$$

Unfortunately, not all matrices are diagonalizable:

Exercise 8) Show that

$$C := \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

is not diagonalizable.



Facts about diagonalizability (see text section 6.2 for complete discussion, with reasoning):

Let  $A_{n \times n}$  have factored characteristic polynomial

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \dots (\lambda - \lambda_m)^{k_m}$$

where like terms have been collected so that each  $\lambda_j$  is distinct (i.e different). Notice that

$$k_1 + k_2 + \dots + k_m = n$$

because the degree of  $p(\lambda)$  is  $n$ .

- Then  $1 \leq \dim(E_{\lambda=\lambda_j}) \leq k_j$ . If  $\dim(E_{\lambda=\lambda_j}) < k_j$  then the  $\lambda_j$  eigenspace is called defective.
- The matrix  $A$  is diagonalizable if and only if each  $\dim(E_{\lambda=\lambda_j}) = k_j$ . In this case, one obtains an  $\mathbb{R}^n$  eigenbasis simply by combining bases for each eigenspace into one collection of  $n$  vectors. (Later on, the same definitions and reasoning will apply to complex eigenvalues and eigenvectors, and a basis of  $\mathbb{C}^n$ .)
- In the special case that  $A$  has  $n$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  each eigenspace is forced to be 1-dimensional since  $k_1 + k_2 + \dots + k_n = n$  so each  $k_j = 1$ . Thus  $A$  is automatically diagonalizable as a special case of the second bullet point.

Exercise 9) How do the examples from today compare with the general facts about diagonalizability?