Math 2250-010
Wed Mar 26

This week's Laplace transform homework is due at 5:00 today.
Exam on Friday in class, 8:05-9:25 a.m.!
Go over last fall exam 2 and answer hw questions, Wed 4:00-5:30 Room TBA (probably here in SW 137)
.
Review for exam in your lab meetings this week.

•   Today we will finish the 10.1-10.3 Laplace transform material (including some table entries from
10.4), that will be covered on the exam Friday. If we've discussed a table entry, you will be expected to be
able to use it. A xerox copy of the book cover Laplace transform table will be provided on the exam. I've
incorporated the left-over material from Monday's notes into today's notes.

•   There are also two topics not on this exam that we skipped earlier, that we should cover today.
    •   forced oscillations in electrical circuits;
    •   using moments of intertia to compute total conserved energy in undamped mechanical
configurations with rotational kinetic energy, and then using this information to find the equation of motion
and natural angular frequency.

Exercise 1) Check why this table entry is true - and notice that it generalizes how the Laplace transforms
of $\cos(kt)$, $\sin(kt)$ are related to those of $e^{at}\cos(kt)$, $e^{at}\sin(kt)$:

$e^{at}f(t)$	$F(s-a)$

Exercise 2) Verify the table entry

$t^n,\ n \in \mathbb{Z}$	$\dfrac{n!}{s^{n+1}}$

by applying one of the results from Monday:
$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n \mathcal{L}\{f(t)\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - ...f^{(n-1)}(0).$$

<u>Exercise 3</u>) Combine 1,2, to get a generalization for the Laplace transform of $t\,e^{a\,t}$, which we also computed on Monday:

$t^n e^{a\,t}$	$\dfrac{n!}{(s-a)^{n+1}}$

<u>Exercise 4</u>) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4\,x(t) = 8\,t\,e^{2\,t}$$
$$x(0) = 0$$
$$x'(0) = 1$$

Exercise 5a) What is the <u>form</u> of the partial fractions decomposition for

$$X(s) = \frac{-356 + 45\,s - 100\,s^2 - 4\,s^5 - 9\,s^4 + 39\,s^3 + s^6}{(s-3)^3\,\left((s+1)^2+4\right)\left(s^2+4\right)^2} \; .$$

5b) Have Maple compute the precise partial fractions decomposition.
5c) What is $x(t) = \mathcal{L}^{-1}\{X(s)\}(t)$?
5d) Have Maple compute the inverse Laplace transform directly.

> $X := s \rightarrow \dfrac{-356 + 45\cdot s - 100\cdot s^2 - 4\cdot s^5 - 9\cdot s^4 + 39\cdot s^3 + s^6}{(s-3)^3\cdot\left((s+1)^2+4\right)\cdot\left(s^2+4\right)^2}$;

> convert(X(s), parfrac, s);
> with(inttrans);
> invlaplace(X(s), s, t);
>

A harder table entry to understand is the following one - go through this computation and see why it seems reasonable, even though there's one step that we don't completely justify. The table entry is

$t f(t)$	$-F'(s)$

Here's how we get it:

$$F(s) = \mathcal{L}\{f(t)\}(s) := \int_0^\infty f(t)e^{-st}\,dt$$

$$\Rightarrow \frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty \frac{d}{ds}f(t)e^{-st}\,dt \ .$$

It's this last step which is true, but needs more justification. We know that the derivative of a sum is the sum of the derivatives, and the integral is a limit of Riemann sums, so this step does at least seem reasonable. The rest is straightforward:

$$\int_0^\infty \frac{d}{ds}f(t)e^{-st}\,dt = \int_0^\infty f(t)(-t)e^{-st}\,dt = -\mathcal{L}\{t f(t)\}(s) \qquad \square.$$

<u>Exercise 6)</u> Use $\mathcal{L}\{t f(t)\}(s) = -F'(s)$ to show the following. (On Monday we used Euler's formula and $\mathcal{L}\{t e^{at}\}(s) = \dfrac{1}{(s-a)^2}$ to get the same results.)

<u>a)</u> $\mathcal{L}\{t\cos(kt)\}(s) = \dfrac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$

<u>b)</u> $\mathcal{L}\left\{\dfrac{1}{2k}t\sin(kt)\right\}(s) = \dfrac{s}{\left(s^2 + k^2\right)^2}$

<u>c)</u> Then use <u>a</u> and the identity

$$\frac{1}{\left(s^2 + k^2\right)^2} = \frac{1}{2k^2}\left(\frac{s^2 + k^2}{\left(s^2 + k^2\right)^2} - \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}\right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2 + k^2\right)^2}\right\}(t) = \frac{1}{2k^2}\left(\frac{1}{k}\sin(kt) - t\cos(kt)\right).$$

<u>Exercise 7a)</u> Use Laplace transforms to write down the solution to

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega_0 t)$$
$$x(0) = x_0$$
$$x'(0) = v_0 .$$

what phenomenon do the solutions to this IVP exhibit?

<u>7b)</u> Use Laplace transforms to solve the general undamped forced oscillation problem, when $\omega \neq \omega_0$:

$$x''(t) + \omega_0^2 x(t) = F_0 \sin(\omega t)$$
$$x(0) = x_0$$
$$x'(0) = v_0$$

$f(t)$, with $|f(t)| \le Ce^{Mt}$	$F(s) := \int_0^\infty f(t)e^{-st}\,dt \quad \text{for } s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1 t t^2 $t^n, \; n \in \mathbb{N}$	$\dfrac{1}{s} \quad (s > 0)$ $\dfrac{1}{s^2}$ $\dfrac{2}{s^3}$ $\dfrac{n!}{s^{n+1}}$	☐ ☐
$e^{\alpha t}$	$\dfrac{1}{s - \alpha} \quad (s > \Re(a))$	☐
$\cos(k\,t)$ $\sin(k\,t)$ $\cosh(k\,t)$ $\sinh(k\,t)$ $e^{a\,t}\cos(k\,t)$ $e^{a\,t}\sin(k\,t)$ $e^{a\,t}f(t)$	$\dfrac{s}{s^2 + k^2} \quad (s > 0)$ $\dfrac{k}{s^2 + k^2} \quad (s > 0)$ $\dfrac{s}{s^2 - k^2} \quad (s > k)$ $\dfrac{k}{s^2 - k^2} \quad (s > k)$ $\dfrac{(s - a)}{(s - a)^2 + k^2} \quad (s > a)$ $\dfrac{k}{(s - a)^2 + k^2} \quad (s > a)$ $F(s - a)$	☐ ☐ ☐ ☐
$f'(t)$ $f''(t)$ $f^{(n)}(t), \; n \in \mathbb{N}$ $\int_0^t f(\tau)\,d\tau$	$s\,F(s) - f(0)$ $s^2 F(s) - s\,f(0) - f'(0)$ $s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\dfrac{F(s)}{s}$	☐ ☐ ☐ ☐
$t\,f(t)$ $t^2 f(t)$ $t^n f(t), \; n \in \mathbb{Z}$ $\dfrac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n F^{(n)}(s)$ $\int_s^\infty F(\sigma)\,d\sigma$	
$t\cos(k\,t)$ $\dfrac{1}{2\,k}t\,\sin(k\,t)$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$ $\dfrac{s}{(s^2 + k^2)^2}$	☐ ☐

$\dfrac{1}{2\,k^3}\left(\sin(k\,t) - k\,t\,\cos(k\,t)\right)$	$\dfrac{1}{(s^2 + k^2)^2}$	
$t\,e^{a\,t}$	$\dfrac{1}{(s-a)^2}$	$\square$
$t^n\,e^{a\,t},\ n \in \mathbb{Z}$	$\dfrac{n!}{(s-a)^{n+1}}$	
more after the midterm!		

Laplace transform table

The mechanical-electrical analogy, continued: Practical resonance is usually bad in mechanical systems, but good in electrical circuits when signal amplification is a goal....recall from earlier in the course:
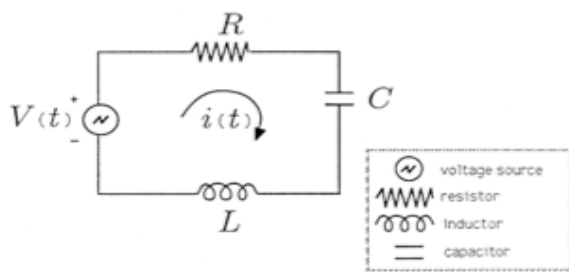
circuit element	voltage drop	units
inductor	$L\,I'(t)$	L Henries (H)
resistor	$R\,I(t)$	R Ohms (Ω)
capacitor	$\dfrac{1}{C}Q(t)$	C Farads (F)

Kirchoff's Law: The sum of the voltage drops around any closed circuit loop equals the applied voltage $V(t)$ (volts).

For $Q(t)$: $\quad L\,Q''(t) + R\,Q'(t) + \dfrac{1}{C}Q(t) = V(t) = E_0 \sin(\omega t)$

For $I(t)$: $\quad L\,I''(t) + R\,I'(t) + \dfrac{1}{C}I(t) = V'(t) = E_0\,\omega \cos(\omega t)$.

Transcribe the work on steady periodic solutions from the preceding pages! The general solution for $I(t)$ is

$$I(t) = I_{sp}(t) + I_{tr}(t) \ .$$

$$I_{sp}(t) = I_0 \cos(\omega t - \alpha) = I_0 \sin(\omega t - \gamma) \ , \quad \gamma = \alpha - \frac{\pi}{2} \ .$$

$$C(\omega) = \frac{F_0}{\sqrt{\left(k - m\,\omega^2\right)^2 + c^2\omega^2}} \quad \Rightarrow I_0(\omega) = \frac{E_0\,\omega}{\sqrt{\left(\dfrac{1}{C} - L\,\omega^2\right)^2 + R^2\,\omega^2}}$$

$$\Rightarrow I_0(\omega) = \frac{E_0}{\sqrt{\left(\dfrac{1}{C\omega} - L\,\omega\right)^2 + R^2}} \ .$$

The denominator $\sqrt{\left(\dfrac{1}{C\omega} - L\,\omega\right)^2 + R^2}$ of $I_0(\omega)$ is called the impedance $Z(\omega)$ of the circuit (because the larger the impedance, the smaller the amplitude of the steady-periodic current that flows through the circuit). Notice that for fixed resistance, the impedance is minimized and the steady periodic current amplitude is maximized when $\dfrac{1}{C\,\omega} = L\,\omega$, i.e.

$$C = \frac{1}{L\,\omega^2} \quad \text{if } L \text{ is fixed and } C \text{ is adjustable (old radios).}$$

$$L = \frac{1}{C\,\omega^2} \quad \text{if } C \text{ is fixed and } L \text{ is adjustable}$$

Both L and C are adjusted in this M.I.T. lab demonstration:
http://www.youtube.com/watch?v=ZYgFuUl9_Vs.

Exercise 8) Multicomponent systems are best understood using conservation of energy, when Newton's law may not apply in any obvious way.   For example, consider the following "rolling mass" configuration (the spring constant of the massless spring is not shown, but as usual we call it k.)
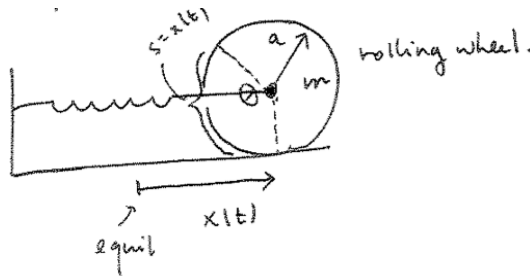
Find the natural angular frequency for the configuration above. Do this by first computing $TE = KE + PE$. Then express the fact that TE is constant once the system is set in motion, by computing its time derivative and setting it equal to the zero function. Use:

(1) The potential energy from stretching/compressing a spring from equilbrium $x = 0$ is the work done in moving it from equilbrium to displacement x, i.e.

$$PE = W = \int_0^x k s \, ds = \frac{1}{2} k x^2.$$

(2) The KE_{trans} from the linear motion (translation) of the disk is $\frac{1}{2} m x'(t)^2$, since $x(t)$ is tracking the center of mass' displacement from equilbrium

(3) The KE_{rot} from the rotation of the disk is given by

$$KE_{rot} = \frac{1}{2} I \omega^2,$$

where $\omega = \theta'(t)$ is the angular frequency of the rotation and I is the moment of inertia, which for a uniform disk of mass m and radius a is given by $I = \frac{1}{2} m a^2$. (Directly computing this rotational KE is a double integral computation, that works out nicely using polar coordinates. You might even have done it in your multivariable calculus class if and when you discussed moments of intertia....in general, moments of intertia are used to compute rotational kinetic energy about centers of mass, and moments are used to compute angular momentum, as well as centers of mass....this is why Calculus classes have units about these topics.)

(The answer is $\omega_0 = \sqrt{\frac{2}{3}} \sqrt{\frac{k}{m}} \approx .82 \sqrt{\frac{k}{m}}$, which is slower than if the mass wasn't rolling.)