Math 2250-010
Mon Mar 24

Go over last fall exam 2, Wednesday 4:00-5:30 Room TBA?

10.1-10.3 Laplace transform, and application to DE IVPs, especially those in Chapter 5. Today we'll continue to fill in the Laplace transform table (on page 2), that we just started using on Friday. Along the way we'll revisit and re-solve many of the mechanical oscillation differential equations from Chapter 5. Today's notes incorporate parts of Friday's, so we won't need to refer back to them.

Exercise 1) (to review) Use the table to compute
1a) $\mathcal{L}\{4 + 2e^{-4t}\}(s)$
1b) $\mathcal{L}^{-1}\left\{\dfrac{2}{s-2} + \dfrac{6}{s}\right\}(t)$.

Exercise 2) Use the definition of Laplace transform and integration by parts to verify that
$$\mathcal{L}\{t\,e^{\alpha t}\}(s) = \frac{1}{(s-\alpha)^2}.$$

As a special case,
$$\mathcal{L}\{t\}(s) = \frac{1}{s^2}.$$

Exercise 3) Use complex number algebra, including Euler's formula, linearity, and the result from Friday that

$$\mathcal{L}\{e^{(a + ki)t}\}(s) = \frac{1}{s - (a + ki)}$$

to verify that

a) $\mathcal{L}\{\cos(kt)\}(s) = \dfrac{s}{s^2 + k^2}$

b) $\mathcal{L}\{\sin(kt)\}(s) = \dfrac{k}{s^2 + k^2}$

c) $\mathcal{L}\{e^{at}\cos(kt)\}(s) = \dfrac{s - a}{(s - a)^2 + k^2}$

d) $\mathcal{L}\{e^{at}\sin(kt)\}(s) = \dfrac{k}{(s - a)^2 + k^2}$.

(Notice that if we tried doing these Laplace transforms directly from the definition, the integrals would be messy but we could attack them via integration by parts or integral tables.)

$f(t)$, with $|f(t)| \le Ce^{Mt}$	$F(s) := \int_0^{\infty} f(t)e^{-st}\, dt$ for $s > M$	$\downarrow$ verified
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$	☐
1	$\dfrac{1}{s}$ $(s > 0)$	☐
t	$\dfrac{1}{s^2}$	
t^2	$\dfrac{2}{s^3}$	
t^n, $n \in \mathbb{N}$	$\dfrac{n!}{s^{n+1}}$	

$e^{\alpha t}$	$\dfrac{1}{s - \alpha}$ $\quad (s > \Re(\alpha))$	☐
$\cos(k\,t)$ $\sin(k\,t)$ $\cosh(k\,t)$ $\sinh(k\,t)$ $e^{a\,t}\cos(k\,t)$ $e^{a\,t}\sin(k\,t)$	$\dfrac{s}{s^2 + k^2}$ $\quad (s > 0)$ $\dfrac{k}{s^2 + k^2}$ $\quad (s > 0)$ $\dfrac{s}{s^2 - k^2}$ $\quad (s > k)$ $\dfrac{k}{s^2 - k^2}$ $\quad (s > k)$ $\dfrac{(s - a)}{(s - a)^2 + k^2}$ $\quad (s > a)$ $\dfrac{k}{(s - a)^2 + k^2}$ $\quad (s > a)$	
$f'(t)$ $f''(t)$ $f^{(n)}(t),\ n \in \mathbb{N}$ $\int_0^t f(\tau)\,d\tau$	$s\,F(s) - f(0)$ $s^2 F(s) - s\,f(0) - f'(0)$ $s^n\,F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$ $\dfrac{F(s)}{s}$	
$t\,f(t)$ $t^2\,f(t)$ $t^n\,f(t),\ n \in \mathbb{Z}$ $\dfrac{f(t)}{t}$	$-F'(s)$ $F''(s)$ $(-1)^n\,F^{(n)}(s)$ $\int_s^{\infty} F(\sigma)\,d\sigma$	
$t\,\cos(k\,t)$ $\dfrac{1}{2\,k}\,t\,\sin(k\,t)$ $\dfrac{1}{2\,k^3}\,(\sin(k\,t) - k\,t\,\cos(k\,t))$	$\dfrac{s^2 - k^2}{(s^2 + k^2)^2}$ $\dfrac{s}{(s^2 + k^2)^2}$ $\dfrac{1}{(s^2 + k^2)^2}$	
$e^{a\,t} f(t)$ $t\,e^{a\,t}$ $t^n\,e^{a\,t},\ n \in \mathbb{Z}$	$F(s - a)$ $\dfrac{1}{(s - a)^2}$ $\dfrac{n!}{(s - a)^{n+1}}$	
more after exam!		

Laplace transform table

Exercise 4) Use integration by parts to show
$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$
Use that identity to show

a) $\mathcal{L}\{f''(t)\}(s) = s^2 F(s) - sf(0) - f'(0),$

b) $\mathcal{L}\{f'''(t)\}(s) = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0),$

c) $\mathcal{L}\{f^{(n)}(t)\}(s) = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0), n \in \mathbb{N}.$

d) $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\}(s) = \dfrac{F(s)}{s}.$

These are the identities that make Laplace transform work so well for initial value problems such as we studied in Chapter 5.

<u>Exercise 5)</u> Use Laplace transforms to solve an IVP we didn't get to on Friday, for an underdamped, unforced oscillator DE. Compare to Chapter 5 method.

$$x''(t) + 6\,x'(t) + 34\,x(t) = 0$$
$$x(0) = 3$$
$$x'(0) = 1$$

Exercise 6) Find $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}(t)$

a) using the result of 4d.
b) using partial fractions.

Exercise 7) Show $\mathcal{L}\{t^n\}(s) = \dfrac{n!}{s^{n+1}}$, $n \in \mathbb{N}$, using the results of 4.

Notice how the Laplace transform table is set up to use partial fraction decompositions for $X(s)$ and linearity in order to find the inverse Laplace transforms $x(t)$. And be amazed at how the table lets you quickly deduce the solutions to important DE IVPs, like we studied in Chapter 5.

Exercise 8) Use Euler's formula and $\mathcal{L}\{t\,e^{\alpha t}\}(s) = \dfrac{1}{(s-\alpha)^2}$ (from Exercise 2) with $\alpha = i\,k$ to show

a) $\mathcal{L}\{t\cos(k\,t)\}(s) = \dfrac{s^2 - k^2}{\left(s^2 + k^2\right)^2}$

b) $\mathcal{L}\left\{\dfrac{1}{2\,k}\,t\sin(k\,t)\right\}(s) = \dfrac{s}{\left(s^2 + k^2\right)^2}$

c) Then use a and the identity

$$\frac{1}{\left(s^2 + k^2\right)^2} = \frac{1}{2\,k^2}\left(\frac{s^2 + k^2}{\left(s^2 + k^2\right)^2} - \frac{s^2 - k^2}{\left(s^2 + k^2\right)^2}\right)$$

to verify the table entry

$$\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2 + k^2\right)^2}\right\}(t) = \frac{1}{2\,k^2}\left(\frac{1}{k}\sin(k\,t) - t\cos(k\,t)\right).$$

Exercise 9) Use Laplace transforms to write down the solution to
$$x''(t) + \omega_0^2\, x(t) = F_0 \sin\left(\omega_0\, t\right)$$
$$x(0) = x_0$$
$$x'(0) = v_0.$$
what phenomena do solutions to this DE illustrate (even though we're forcing with $\sin\left(\omega_0 t\right)$ rather than $\cos\left(\omega_0 t\right)$)? How would you have tried to solve this problem in Chapter 5?

Exercise 10) Solve the following IVP. Use this example to recall the general partial fractions algorithm.

$$x''(t) + 4\,x(t) = 8\,t\,e^{2\,t}$$
$$x(0) = 0$$
$$x'(0) = 1$$