Math 2250-010
Mon Mar 17

Section 5.5: Finding y_P for non-homogeneous linear differential equations

$$L(y) = f$$

(so that you can use the general solution $y = y_P + y_H$ to solve initial value problems, and because sometimes a good choice for y_P contains the most essential information in dynamical systems problems).

There are two methods we will use:

- The method of undetermined coefficients uses guessing algorithms, and works for constant coefficient linear differential equations with certain classes of functions $f(x)$ for the non-homogeneous term. The method seems magic, but actually relies on vector space theory. We've already seen simple examples of this, where we seemed to pick particular solutions out of the air. This method is the main focus of section 5.5.

- The method of variation of parameters is more general, and yields an integral formula for a particular solution y_P, assuming you are already in possession of a basis for the homogeneous solution space. This method has the advantage that it works for any linear differential equation and any (continuous) function f. It has the disadvantage that the formulas can get computationally messy especially for differential equations of order $n > 2$. We'll study the case $n = 2$ only.

The easiest way to explain the method of undetermined coefficients is with examples.

Roughly speaking, you make a "guess" with free parameters (undetermined coefficients) that "looks like" the right side. AND, you need to include all possible terms in your guess that could arise when you apply L to the terms you know you want to include.

We'll make this more precise later in today's notes.

Exercise 1) Find a particular solution $y_P(x)$ for the differential equation

$$L(y) := y'' + 4y' - 5y = 10x + 3 .$$

Hint: try $y_P(x) = d_1 x + d_2$ because L transforms such functions into ones of the form $b_1 x + b_2$. d_1, d_2 are your "undetermined coefficients", for the given right hand side coefficients $b_1 = 10, b_2 = 3$.

Exercise 2) Use your work in 1 and your expertise with homogeneous linear differential equations to find the general solution to

$$y'' + 4\,y' - 5\,y = 10\,x + 3$$

Exercise 3) Find a particular solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x}.$$

Hint: try $y_P = d\,e^{2\,x}$ because L transforms functions of that form into ones of the form $b\,e^{2\,x}$, i.e. $L\left(d\,e^{2\,x}\right) = b\,e^{2\,x}$. "$d$" is your "undetermined coefficient" for $b = 14$.

Exercise 4a) Use superposition (linearity of the operator L) and your work from the previous exercises to find the general solution to

$$L(y) = y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6\,.$$

4b) Solve (or at least set up the problem to solve) the initial value problem

$$y'' + 4\,y' - 5\,y = 14\,e^{2\,x} - 20\,x - 6$$
$$y(0) = 4$$
$$y'(0) = -4\,.$$

4c) Check your answer with technology.

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> with(DEtools) :
> dsolve({y''(x) + 4·y'(x) − 5·y(x) = 14·e^(2·x) − 20·x − 6, y(0) = 4, y'(0) = -4});
>
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Exercise 5) Find a particular solution to

$$L(y) := y'' + 4y' - 5y = 2\cos(3x) .$$

Hint: To solve $L(y) = f$ we hope that f is in some finite dimensional subspace V that is preserved by L, i.e. $L : V \rightarrow V$.

- In Exercise 1 $V = span\{1, x\}$ and so we guessed $y_P = d_1 + d_2 x$.
- In Exercise 3 $V = span\{e^{2x}\}$ and so we guessed $y_P = d e^{2x}$.
- What's the smallest subspace V we can take in the current exercise? Can you see why

$V = span\{\cos(3x)\}$ and a guess of $y_P = d\cos(3x)$ won't work?

All of the previous exercises rely on:

Method of undetermined coefficients (base case): If you wish to find a particular solution y_P, i.e. $L(y_P) = f$ and if the non-homogeneous term f is in a finite dimensional subspace V with the properties that
(i) $L : V \rightarrow V$, i.e. L transforms functions in V into functions which are also in V; and
(ii) The only function $g \in V$ for which $L(g) = 0$ is $g = 0$.

Then there is always a unique $y_P \in V$ with $L(y_P) = f$.

why: We already know this fact is true for matrix transformations $L(\underline{x}) = A_{n \times n}\underline{x}$ with $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (because if the only homogeneous solution is $\underline{x} = \underline{0}$ then A reduces to the identity, so also each matrix equation $A\,\underline{x} = \underline{b}$ has a unique solution $\underline{x}$. The theorem above is a generalization of that fact to general linear transformations. There is an "appendix" explaining this at the end of today's notes, for students who'd like to understand the details.

Exercise 6) Use the method of undetermined coefficients to guess the form for a particular solution $y_P(x)$ for a constant coefficient differential equation

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

(assuming the only such solution in your specified subspace V that would solve the homogeneous DE is the zero solution):

6a) $L(y) = x^3 + 6x - 5$

6b) $L(y) = 4 e^{2x} \sin(3x)$

6c) $L(y) = x \cos(2x)$

BUT LOOK OUT

Exercise 7a) Find a particular solution to

$$y'' + 4y' - 5y = 4e^x .$$

Hint: since $p(r) = r^2 + 4r - 5 = (r-1)(r+5)$ and $y_H = c_1 e^x + c_2 e^{-5x}$, a guess of $y_P = a\, e^x$ will not work (and $span\{e^x\}$ does not satisfy the "base case" conditions for undetermined coefficients). Instead try

$$y_P = d\, x\, e^x$$

and factor $L = D^2 + 4D - 5I = (D + 5I) \circ (D - I)$.

7b) check work with technology

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> with(DEtools) :
> dsolve(y''(x) + 4·y'(x) − 5·y(x) = 4·e^x, y(x));
>
```

A vector space theorem like the one for the base case, except for $L : V \rightarrow W$, combined with our understanding of how to factor constant coefficient differential operators (as in last week's homework) leads to an extension of the method of undetermined coefficients, for right hand sides which can be written as sums of functions having the indicated forms below. See the discussion in section 5.5 of the text, pages 369-374 of the new edition of our text, and the table 5.5.1, reproduced here.

Method of undetermined coefficients (extended case): If L has a factor $(D - r\,I)^s$ then the guesses you would have made in the "base case" need to be multiplied by x^s , as in Exercise 7. (If you understood the homework problem last week about factoring L into composition of terms like $(D - r\,I)^s$, then you have an inkling of why this recipe works.)

See discussion p 341-346. Also p.346 Table: Don't forget to also use superposition for $f_1 + f_2$

$f(x)$	y_p	potential related root of $p(r)$
$P_m(x) = b_0 + b_1 x + \dots + b_m x^m$	$x^s(A_0 + A_1 x + A_2 x^2 + \dots + A_m x^m)$	$r = 0$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$	$r = ik$
$e^{rx}(a\cos kx + b\sin kx)$	$x^s e^{rx}(A\cos kx + B\sin kx)$	root $= r + ik$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1 x + \dots + A_m x^m)e^{rx}$	root $= r$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1 x + \dots + A_m x^m)\cos kx + (B_0 + B_1 x + \dots + B_m x^m)\sin kx]$	root $= ik$

$x^s = 1$ $(s = 0)$ as long as the potential root is not actually a root of the characteristic polynomial for L (easy case, Exercises 1-4)

Otherwise, s is the power of (r-root) in factored $p(r)$

e.g. in Exercise 5,

$p(r) = (r+5)^1(r-1)^1$ ← so multiply guess by x^1.

Exercise 8) Set up the undetermined coefficients particular solutions for the examples below. When necessary use the extended case to modify the undetermined coefficients form for y_P. Use technology to check if your "guess" form was right.

$$L(y) := y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f$$

8a) $y''' + 2y'' = x^2 + 6x$
(So the characteristic polynomial for $L(y) = 0$ is $r^3 + 2r^2 = r^2(r+2) = (r-0)^2(r+2)$.)

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> dsolve(y'''(x) + 2·y''(x) = x^2 + 6·x, y(x));
>
```

8b) $y'' - 4y' + 13y = 4e^{2x}\sin(3x)$
(So the characteristic polynomial for $L(y) = 0$ is
$r^2 - 4r + 13) = (r-2)^2 + 9 = (r-2+3i)(r-2-3i)$.)

```
> dsolve(y''(x) - 4·y'(x) + 13·y(x) = 4·e^(2·x)·sin(3·x), y(x));
>
```

Not every right hand side is amenable to finding particular solutions via undetermined coefficients. Luckily there is a more general (but technically messier) way that will always work:

Variation of Parameters: The advantage of this method is that is always provides a particular solution, even for non-homogeneous problems in which the right-hand side doesn't fit into a nice finite dimensional subspace preserved by L, and even if the linear operator L is not constant-coefficient. The formula for the particular solutions can be somewhat messy to work with, however, once you start computing.

Here's the formula: Let $y_1(x), y_2(x), \ldots y_n(x)$ be a basis of solutions to the homogeneous DE

$$L(y) := y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0\,.$$

Then $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \ldots + u_n(x)y_n(x)$ is a particular solution to

$$L(y) = f$$

provided the coefficient functions (aka "varying parameters") $u_1(x), u_2(x), \ldots u_n(x)$ have derivatives satisfying the matrix equation

$$\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \left[W(y_1, y_2, \ldots, y_n)\right]^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ f \end{bmatrix}$$

where $\left[W(y_1, y_2, \ldots, y_n)\right]$ is the Wronskian matrix.

Here's how to check this fact when $n = 2$: Write

$$y_p = y = u_1y_1 + u_2y_2\,.$$

Thus

$$y' = u_1y_1' + u_2y_2' + \left(u_1'y_1 + u_2'y_2\right).$$

Set

$$\left(u_1'y_1 + u_2'y_2\right) = 0.$$

Then

$$y'' = u_1y_1'' + u_2y_2'' + \left(u_1'y_1' + u_2'y_2'\right).$$

Set

$$\left(u_1'y_1' + u_2'y_2'\right) = f.$$

Notice that the two $(\ldots)$ equations are equivalent to the matrix equation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

which is equivalent to the $n = 2$ version of the claimed condition for y_p. Under these conditions we compute

$$\begin{aligned} &p_0\left[y = u_1y_1 + u_2y_2\right] \\ &+ p_1\left[y' = u_1y_1' + u_2y_2'\right] \\ &+ 1\left[y'' = u_1y_1'' + u_2y_2'' + f\right] \\ &\quad L(y) = u_1L(y_1) + u_2L(y_2) + f \\ &L(y) = 0 + 0 + f = f \end{aligned}$$

Exercise 9) Rework Exercise 7a with variation of parameters, i.e. find a particular solution to

$$y'' + 4\,y' - 5\,y = 4\,e^{x}$$

of the form

$$y_P(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = u_1 e^{x} + u_2 e^{-5\,x}.$$

Appendix: The following two theorems justify the method of undetermined coefficients, in both the "base case" and the "extended case." We will not discuss these in class, but I'll be happy to chat about the arguments with anyone who's interested, outside of class. They only use ideas we've talked about already, although they are abstract.

Theorem 0:

- Let V and W be vector spaces. Let V have dimension $n < \infty$ and let $\{\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\boldsymbol{v}}_n\}$ be a basis for V.
- Let $L : V \rightarrow W$ be a linear transformation, i.e. $L(\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}) = L(\underline{\boldsymbol{u}}) + L(\underline{\boldsymbol{v}})$ and $L(c\,\underline{\boldsymbol{u}}) = c\,L(\underline{\boldsymbol{u}})$ holds $\forall\ \underline{\boldsymbol{u}}, \underline{\boldsymbol{v}} \in V, c \in \mathbb{R}$.) Consider the range of L, i.e.

$$\begin{aligned} Range(L) &:= \{L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n) \in W, \text{such that each } d_j \in \mathbb{R}\} \\ &= \{d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\boldsymbol{v}}_n) \in W, \text{such that each } d_j \in \mathbb{R}\} \\ &= span\{L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ...\, L(\underline{\boldsymbol{v}}_n)\}. \end{aligned}$$

Then $Range(L)$ is $n - dimensional$ if and only if the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ is $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$.

proof:

(i) $\Leftarrow$: The only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ is $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$ implies $Range(L)$ is $n - dimensional$:

If we can show $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ...\, L(\underline{\boldsymbol{v}}_n)$ are linearly independent, then they will be a basis for $Range(L)$ and this subspace will have dimension n. So, consider the dependency equation:

$$d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\boldsymbol{v}}_n) = \underline{\mathbf{0}}.$$

Because L is a linear transformation, we can rewrite this equation as

$$L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n) = \underline{\mathbf{0}}.$$

Under our assumption that the only homogeneous solution is the zero vector, we deduce

$$d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n = \underline{\mathbf{0}}.$$

Since $\underline{\boldsymbol{v}}_1, \underline{\boldsymbol{v}}_2, ..., \underline{\boldsymbol{v}}_n$ are a basis they are linearly independent, so $d_1 = d_2 = ... = d_n = 0$.

□

(ii) $\Rightarrow$: $Range(L)$ is $n - dimensional$ implies the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ is $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$: Since the range of L is $n - dimensional$, $L(\underline{\boldsymbol{v}}_1), L(\underline{\boldsymbol{v}}_2), ...\, L(\underline{\boldsymbol{v}}_n)$ must be linearly independent. Now, let $\underline{\boldsymbol{v}} = d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n$ be a homogeneous solution, $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$. In other words,

$$\begin{aligned} &L(d_1\underline{\boldsymbol{v}}_1 + d_2\underline{\boldsymbol{v}}_2 + ... + d_n\underline{\boldsymbol{v}}_n) = \underline{\mathbf{0}} \\ \Rightarrow\ &d_1L(\underline{\boldsymbol{v}}_1) + d_2L(\underline{\boldsymbol{v}}_2) + ... + d_nL(\underline{\boldsymbol{v}}_n) = \underline{\mathbf{0}} \\ \Rightarrow\ &d_1 = d_2 = ... = d_n = 0 \Rightarrow \underline{\boldsymbol{v}} = \underline{\mathbf{0}}. \end{aligned}$$

□

Theorem 1 Let Let V and W be vector spaces, both with the same dimension $n < \infty$. Let $L : V \rightarrow W$ be a linear transformation. Let the only solution to $L(\underline{\boldsymbol{v}}) = \underline{\mathbf{0}}$ be $\underline{\boldsymbol{v}} = \underline{\mathbf{0}}$. Then for each $\underline{\boldsymbol{w}} \in W$ there is a unique $\underline{\boldsymbol{v}} \in V$ with $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$.

proof: By Theorem 0, the dimension of $Range(L)$ is $n - dimensional$. Therefore it must be all of W. So for each $\underline{\boldsymbol{w}} \in W$ there is at least one $\boldsymbol{v}_P \in V$ with $L(\underline{\boldsymbol{v}}_P) = \underline{\boldsymbol{w}}$. But the general solution to $L(\underline{\boldsymbol{v}}) = \underline{\boldsymbol{w}}$ is $\underline{\boldsymbol{v}} = \underline{\boldsymbol{v}}_P + \underline{\boldsymbol{v}}_H$, where $\underline{\boldsymbol{v}}_H$ is the general solution to the homogeneous equation. By assumption, $\underline{\boldsymbol{v}}_H = \underline{\mathbf{0}}$, so the particular solution is unique.

□

Remark: In the base case of undetermined coefficients, $W = V$. In the extended case, W is the space in which f lies, and $V = x^s W$, i.e. the space of all functions which are obtained from ones in W by multiplying them by x^s. This is because if L factors as

$$L = (D - r_1 I)^{k_1} \circ (D - r_2 I)^{k_2} \circ \ldots \circ (D - r_m I)^{k_m}$$

and if f is in a subspace W associated with the characteristic polynomial root r_m, then for $s = k_m$ the factor $(D - r_m I)^{k_m}$ of L will transform the space $V = x^s W$ back into W, and not transform any non-zero function in V into the zero function. And the other factors of L will then preserve W, also without transforming any non-zero elements to the zero function.