

Part 1: This is part of the homework that is due on Friday at the start of class.

1. (a) (this is w8.1b) Find the general solution to the differential equation for $x(t)$

$$x''(t) + 4x'(t) + 29x(t) = 0.$$

Hint: completing the square works well here - probably better than the quadratic formula.

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- (b) (this is w8.1d) Find the general solution to the differential equation for $y(x)$

$$y^{(4)} - 8y' = 0.$$

1. (10 points) The figure below depicts a typical mass-spring-dashpot system; here, m denotes the mass of the oscillating object (in kilograms), k denotes the spring constant (in Newtons/meter), and c denotes the damping constant. We use $x(t)$ to denote the position of the mass at time t (with $x = 0$ being its equilibrium position); positive directions of x imply the mass is to the right of its equilibrium position. Throughout, we fix the values $m = 3$ and $k = 48$. However, **the value of the damping constant c will vary**. In this setup, $x(t)$ satisfies the ODE $3x''(t) + cx'(t) + 48x(t) = 0$, which after dividing through by the mass 3 is equivalent to

$$x''(t) + \frac{c}{3} \cdot x'(t) + 16 \cdot x(t) = 0$$

Throughout this problem, “IVP” will refer to this ODE subject to the initial conditions

$$x(0) = 3 \text{ and } x'(0) = -4$$

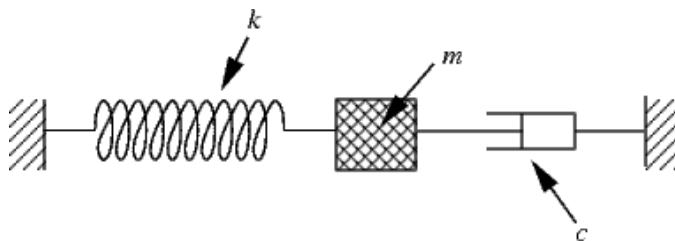


Figure 1:

- (a) Solve the IVP when there is no damping, i.e., when $c = 0$. After finding your solution in linear combination form, convert into amplitude-phase form. Identify the numerical values for amplitude, angular frequency, frequency, period, phase angle, and time delay.

- (b) Solve the IVP when $c = 1.2$ (we refer to the situation when c is small enough so that the characteristic polynomial has complex roots as *under-damped*). **Instructions:** When finding the roots of the characteristic polynomial round off to five digits.
- (c) Recall that a value of the damping constant induces so-called *critical damping* whenever the associated characteristic polynomial has a double real root. Confirm that $c = 24$ leads to critical damping, and solve the IVP in this case.
- (d) Finally, solve the IVP when $c = 51$ (here, we are in the situation of *over-damping*).
- (e) Use Maple or other software (Matlab, Wolfram alpha, etc) to create a display containing the graphs of all four solutions above, on the interval $0 \leq t \leq 4$. Print out a copy, and label which graph corresponds to which solution.

2. (10 points) Energy in a mass-spring-damper system: Let $x(t)$ be the position of a mass m attached to a spring with Hooke's constant k and damping piston with constant c , yielding the differential equation

$$mx'' + cx' + kx = f(t),$$

where $f(t)$ is an external forcing on the mass. We wish to account for the total energy of the mass-spring configuration, neglecting the heat energy loss due to damping. We define the total energy $E(t)$ to be the sum of kinetic and potential energy. Potential energy $PE(t)$ is stored by the compressed or stretched spring, and is the work done to stretch/compress it as the mass moves from equilibrium $x = 0$, to position x :

$$PE(t) = \int_0^x kudu = \frac{k}{2}x^2$$

As usual, the kinetic energy of the mass is

$$KE(t) = \frac{m}{2}(x')^2.$$

The sum is the total energy

$$E(t) = PE(t) + KE(t) = \frac{1}{2}(kx^2 + m(x')^2)$$

- (a) Take the derivative of $E(t)$ with respect to time. Use the chain rule on the right side of the equation. Then simplify your result, so that you get a formula for $E'(t)$ that only depends on the forcing $f(t)$, the velocity $x'(t)$, and the damping coefficient c .

- (b) Assume there is no external forcing, i.e. $f = 0$. In this case, what condition(s) on m, c, k guarantee that the energy in the system is constant (i.e., $dE/dt = 0$)?
- (c) Set $f(t) = 0$, $m = 1$, $c = 2$, and $k = 5$. Set initial conditions to be $x(0) = 2$ and $x'(0) = 0$. How long will it take for the system to lose 80% of its initial energy? To solve, find the solution $x(t)$ to the DE and use this solution to compute the energy function E explicitly.
- (d) Plot the energy curve $E(t)$ and describe its behavior. Explain in words the mechanism that drives the energy picture that you observe, and why the energy is not decreasing initially, but then decreases more rapidly later on.