

Part 1: This is part of the homework that is due on Friday at the start of class.

1. (This is homework problem w7.1.) Consider the three vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

- (a) Compute the magnitudes (lengths),  $|\mathbf{u}|$  and  $|\mathbf{v}|$ .
- (b) Express  $\mathbf{w}$  as a linear combination of  $\mathbf{u}$  and  $\mathbf{v}$ , by solving the appropriate linear system.
- (c) Make a careful and accurate sketch which illustrates your answer to (b).
- (d) Find a linear combination of  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  which adds up to zero vector. Thus these three vectors are *linearly dependent*.

2. (This is homework problem w7.2) Consider the three vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ -5 \end{bmatrix}$$

- (a) Use a reduced row echelon computation to check that the span of these three vectors is not  $\mathbf{R}^3$ .
- (b) Use your reduced row echelon form computation to write  $\mathbf{w}$  as a linear combination of  $\mathbf{u}, \mathbf{v}$ . (Hint: what augmented matrix would you have if you were solving  $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{w}$  for  $c_1, c_2$ ?)
- (c) The span of these three vectors is actually a plane through the origin. Every plane through the origin can be defined by an implicit equation  $ax + by + cz = 0$  with some constants  $a, b, c$ . Find the constants  $a, b, c$  so that all points  $(x, y, z)$  whose position vectors  $[x, y, z]^T$  are in the span of  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  lie in the plane given by  $ax + by + cz = 0$ .

3. (This is homework problem w7.3.) Consider the three vectors

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- (a) Compute the reduced row echelon form of the matrix  $\langle \mathbf{u} | \mathbf{v} | \mathbf{w} \rangle$  to deduce that every vector  $\mathbf{b} \in \mathbf{R}^3$  can be expressed as a linear combination of these three vectors, i.e

$$c_1 \mathbf{u} + c_2 \mathbf{v} + c_3 \mathbf{w} = \mathbf{b}$$

is uniquely solvable for the linear combination coefficients  $c_1, c_2, c_3$ . (In particular, the only linear combination that can yield the zero vector is when  $c_1 = c_2 = c_3 = 0$ , so  $\langle \mathbf{u} | \mathbf{v} | \mathbf{w} \rangle$  are also linearly independent.) Since these three vectors span  $\mathbf{R}^3$  and are linearly independent, we call them a basis for  $\mathbf{R}^3$ .

- (b) Express  $\mathbf{v} = [5, -7, 5]^T$  as a unique linear combination of these basis vectors.
- (c) Compute the determinant of the matrix you worked with in part (a), and explain why the result of (a) also follows from this computation.

1. (10 points) Recall, that the dot product of two same-length vectors is computed algebraically as the sum of the products of the corresponding entries. In physics and multivariable calculus one learns that the dot product contains geometric information, via the identity

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta,$$

where  $\theta$  is the angle between  $\mathbf{u}, \mathbf{v}$ . (See the extra credit problem at the end of this lab, to review why this geometric identity is true.)

- (a) As a special case of the identity above, explain why non-zero vectors  $\mathbf{u}$  and  $\mathbf{v}$  are perpendicular exactly when  $\mathbf{u} \cdot \mathbf{v} = 0$ .
- (b) Let  $W$  be a plane with  $(x_0, y_0, z_0) \in W$ . Let  $(x, y, z)$  be any other point in the plane. Let vector  $(a, b, c)^T$  be perpendicular to  $W$  (a normal vector). Explain why  $x, y, z$  must satisfy the “plane” equation:

$$ax + by + cz = d$$

where  $d = ax_0 + by_0 + cz_0$ . Hint: the vector going from  $(x_0, y_0, z_0)$  to  $(x, y, z)$  must be perpendicular to the normal vector, allowing you to use the dot product identity when  $\cos(\theta) = 0$ .

- (c) For the plane through the origin given by  $2x + 5y - z = 0$ , verify the three position vectors  $\mathbf{u} = [1, 0, 2]^T$ ,  $\mathbf{v} = [-3, 2, 4]^T$ ,  $\mathbf{w} = [0, 2, 10]^T$  (from the origin, which is in the plane, to the endpoints of the position vectors, which are also in the plane) are perpendicular to the normal vector  $[2, 5, -1]^T$ .
- (d) In multivariable calculus and physics you learn how to use the cross product of pairs of vectors in 3-space, in order to find perpendicular vectors. Take the cross product of the vectors  $\mathbf{u}, \mathbf{v}$  in part (c), and verify that you get a multiple of the normal vector  $[2, 5, -1]^T$ .

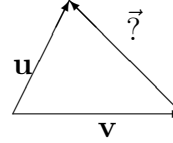
2. (10 points) (a) Suppose we have a matrix  $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$  and a square determined by points  $(0, 0), (0, 1), (1, 1), (1, 0)$ . What will the image of the square look like under transformation by the matrix  $A$ ? i.e. let

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where  $(x_1, x_2)^T$  is a point on the original square and  $(y_1, y_2)^T$  is its image after transformation by  $A$ . Draw the image of the square after this transformation.

- (b) Find the area of the image parallelogram in (a). (Hint: think of the parallelogram as sitting inside a larger rectangle  $0 \leq y_1 \leq 4, 0 \leq y_2 \leq 4$ ). Compare this area to the determinant of  $A$ .
- (c) Show, for the general  $\mathbf{u}, \mathbf{v}$  in the first quadrant, with  $\mathbf{v}$  counterclockwise from  $\mathbf{u}$ , the area of the parallelogram having the vectors  $\mathbf{u}, \mathbf{v}$  as adjacent sides (as in the special case above) always equals  $\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$ .
- (d) What happens if you compute  $\begin{vmatrix} v_1 & u_1 \\ v_2 & u_2 \end{vmatrix}$  instead?

3. (10 points) (Extra credit - not required.) Given two vectors  $\mathbf{u}, \mathbf{v}$ , that are not scalar multiples, i.e  $\mathbf{u} \neq c\mathbf{v}$ , consider the diagram below. We will assume that the vectors have been labeled so that  $|\mathbf{v}| > |\mathbf{u}|$ , and that the angle  $\theta$  between  $\mathbf{u}, \mathbf{v}$  is an angle of at most  $\frac{\pi}{2}$ .



- (a) Show  $\vec{?} = \mathbf{u} - \mathbf{v}$  using vector addition.
- (b) Compute  $|\mathbf{u} - \mathbf{v}|^2$  in two ways.
  - i. by expanding dot product identity  $|\mathbf{u} - \mathbf{v}|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$  (which is true because the magnitude squared of any vector is the dot product of that vector with itself).
  - ii. by using trigonometry and the Pythagorean Theorem on two subtriangles that have right angles - obtained by drawing a line segment perpendicular to  $\mathbf{v}$ , from the vertex where  $\mathbf{u}$  and  $\vec{?}$  meet, down to  $\mathbf{v}$ .
- (c) Use your work in (b) to deduce that  $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos \theta$  where  $\theta$  is the acute angle between  $\mathbf{u}$  and  $\mathbf{v}$ .