Math 2250-010
Monday January 6

- Notes will often contain some material that we will not get to until the following lecture. That is likely the case today.

- Go over course information on syllabus and course homepage:

http://www.math.utah.edu/~korevaar/2250spring14

- Notice that there is homework due this Friday, and that your first lab meetings are this week.

Then, let's begin!

Section 1.1 Introduction to differential equations

- What is an n^{th} order differential equation (DE)?

any equation involving a function $y = y(x)$ and its derivatives, for which the highest derivative appearing in the equation is the n^{th} one, $y^{(n)}(x)$; i.e. any equation which can be written as

$$F\left(x, y(x), y'(x), y''(x), \ldots y^{(n)}(x)\right) = 0.$$

Exercise 1: Which of the following are differential equations? For each DE determine the order.

a) For $y = y(x)$, $\left(y''(x)\right)^2 + \sin(y(x)) = 0$

b) For $x = x(t)$, $x'(t) = 3\,x(t)(10 - x(t))$.

c) For $x = x(t)$, $x' = 3\,x\,(10 - x)$.

d) For $z = z(r)$, $z'''(r) + 4\,z(r)$.

e) For $y = y(x)$, $y' = y^2$.

Definitions:

- A function $y(x)$ solves the differential equation $F(x, y, y', y'', y^{(n)}) = 0$ on some interval I (or is a solution function for the differential equation) means that $y(x)$ makes the differential equation a true equality for all x in I.

- A 1^{st} order DE is an equation involving a function and its first derivative. We may chose to write the function and variable as $y = y(x)$. In this case the differential equation is an equation equivalent to one of the form

$$F(x, y, y') = 0 .$$

Chapters 1-2 are about first order differential equations. For first order differential equations as above we can often use algebra to solve for y' in order to get what we call the **standard form** for the first order DE:

$$y' = f(x, y) .$$

- If we want our solution function to a first order DE to also satisfy $y(x_0) = y_0$, and if our DE is written in standard form, then we say that we are studying an **initial value problem** (IVP):

$$y' = f(x, y)$$
$$y(x_0) = y_0 .$$

If we can find a solution function $y(x)$ that makes both equations of the the initial value problem true, then we say that $y(x)$ solves the initial value problem.

Exercise 2: Consider the differential equation $\frac{dy}{dx} = y^2$ from (1e).

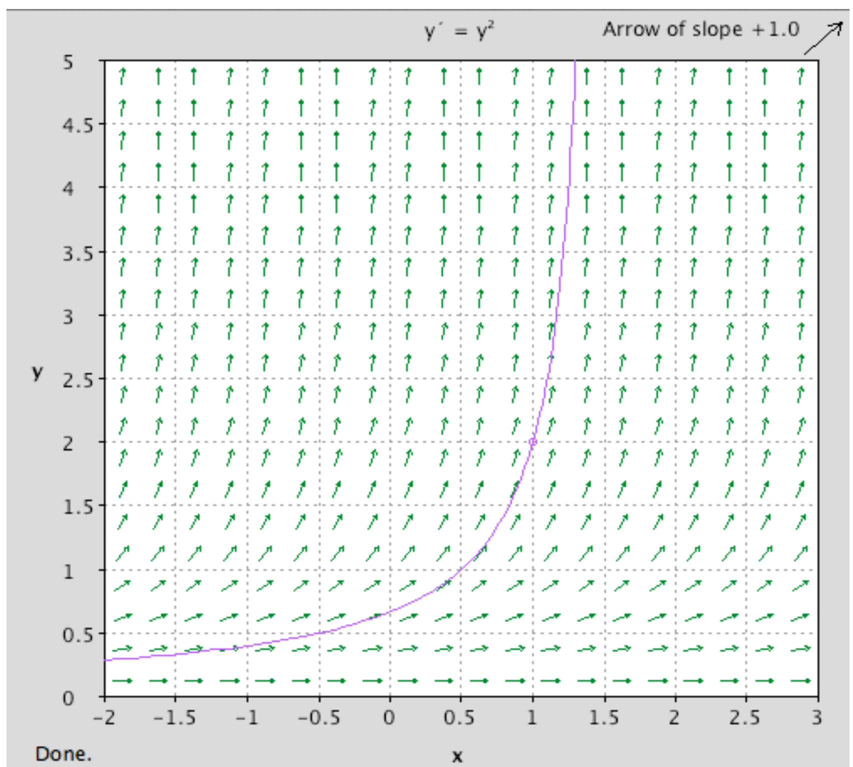
a) Show that functions $y(x) = \frac{1}{C - x}$ solve the DE (on any interval not containing the constant C).

b) Find the appropriate value of C to solve the initial value problem

$$y' = y^2$$
$$y(1) = 2 .$$

2c) What is the largest interval on which your solution to 2b is defined as a differentiable function? Why?

2d) Do you expect that there are any other solutions to the IVP in 2b? Hint: The graph of the IVP solution function we found is superimposed onto a "slope field" below, where the line segment slopes at points (x, y) have values y^2 (because solution graphs to our differential equation will have those slopes, according to the differential equation). This might give you some intuition about whether you expect more than one solution to the IVP.



• **important course goals:** understand some of the key differential equations which arise in modeling real-world dynamical systems from science, mathematics, engineering; how to find the solutions to these differential equations if possible; how to understand properties of the solution functions (sometimes even without formulas for the solutions) in order to effectively model or to test models for dynamical systems.

In fact, you've encountered differential equations in previous mathematics and/or physics classes. For example:

• 1^{st} order differential equations: rate of change of function depends in some way on the function value, the variable value, and nothing else. For example, you've studied the population growth/decay differential equation for $P = P(t)$, and k a constant, given by

$$P'(t) = k\,P(t)$$

and having applications in biology, physics, finance. In this model, how fast the "population" changes is proportional to the population.

• 2^{nd} order DE's: Newton's second law (change in momentum equals net forces) often leads to second order differential equations for particle position functions $x = x(t)$ in physics.

<u>Exercise 3</u>: The mathematical model in which the time rate of change of a population $P(t)$ is proportional to that population is expressed mathematically as

$$\frac{dP}{dt} = k\,P$$

where k is the proportionality constant.

<u>3a)</u> Find all solutions to this differential equation by using the chain rule backwards.

<u>3b)</u> The method of "separation of variables" is taught in most Calc I courses, and we'll cover it in detail in section 1.4. It's an algorithm which hides the "chain rule backwards" technique by treating the derivative $\frac{dP}{dt}$ as a quotient of differentials. Recall this magic algorithm to recover the solutions from <u>3a</u>.

Exercise 4) Newton's law of cooling is a model for how objects are heated or cooled by the temperature of an ambient medium surrounding them. In this model, the body temperature $T = T(t)$ changes at a rate proportional to to the difference between it and the ambient temperature $A(t)$. In the simplest models A is constant.

a) Use this model to derive the differential equation

$$\frac{dT}{dt} = -k(T - A) .$$

b) Would the model have been correct if we wrote $\frac{dT}{dt} = k(T - A)$ instead?

c) Use this model to partially solve a murder mystery: At 3:00 p.m. a deceased body is found. Its temperature is 70° F. An hour later the body temperature has decreased to 60°. It's been a winter inversion in SLC, with constant ambient temperature 30°. Assuming the Newton's law model, estimate the time of death.

Section 1.2: differential equations equivalent to ones of the form $y'(x) = f(x)$, which we solve by direct antidifferentiation.

An important class of such problems arises in physics, usually as velocity/acceleration problems via Newton's second law. Recall that if a particle is moving along a number line and if $x(t)$ is the particle **position** function at time t, then the rate of change of $x(t)$ (with respect to t) namely $x'(t)$, is the **velocity** function. If we write $x'(t) = v(t)$ then the rate of change of velocity $v(t)$, namely $v'(t)$, is called the **acceleration** function $a(t)$, i.e.

$$x''(t) = v'(t) = a(t) .$$

Thus if $a(t)$ is known, e.g. from Newton's second law that force equals mass times acceleration, then one can antidifferentiate once to find velocity, and one more time to find position.

Exercise 5:

a) If the units for position are meters m and the units for time are seconds s , what are the units for velocity and acceleration? (These are *mks* units.)

b) Same question, if we use the English system in which length is measured in feet and time in seconds. Could you convert between *mks* units and English units?

Exercise 6:

Suppose the acceleration function is a negative constant $-a$,

$$x''(t) = -a .$$

(This could happen for vertical motion, e.g. near the earth's surface with $a = g \approx 9.8 \frac{m}{s^2} \approx 32 \frac{ft}{s^2}$, as well as in other situations.)

a) Write $x(0) = x_0$, $v(0) = v_0$ for the initial position and velocity. Find formulas for $v(t)$ and $x(t)$.

b) Assuming $x(0) = 0$ and $v_0 > 0$, show that the maximum value of $x(t)$ is

$$x_{\max} = \frac{1}{2}\frac{v_0^2}{a} .$$

(This formula may help with some homework problems, as well as with the next exercise.)

Exercise 7: Car accident reconstruction. A driver skids 210 ft. after applying his brakes. He claims to the investigating officers that he was going 25 miles per hour before trying to stop. A police test of his vehicle shows that if the brakes are applied to force a skid at an initial speed of 25 $\frac{mi}{h}$ then the auto skids only 45 ft. Assuming that the car is decelerating at a constant rate while skidding, about how fast was the driver really going?