Math 2250-010

Fri Jan 10

1.3-1.4   more slope fields; existence and uniqueness for solutions to IVPs; using separable differential equations for examples.

- Quiz at the start of class, on sections 1.1-1.2.

- Discuss the exercises on Wednesday's notes that are applications from section 1.2. Because the two lab assignments cover two of the exercises, we will only discuss Exercise 2, the swimmer's problem, in depth. There are a lot of important mathematical ideas from Calculus hidden in that application, and we will review/discuss these.

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Section 1.3: slope fields and graphs of solutions to first order differential equations

Consider an initial value problem for a function $y(x)$:

$$y' = f(x, y)$$
$$y(x_0) = y_0 \; .$$

If $y(x)$ is a solution to this IVP and if we consider its graph $y = y(x)$ in the $x - y$ plane, then the initial condition means the graph must pass through the point (x_0, y_0). Because $y(x)$ solves the differential equation, at every point (x, y) on the graph $y = y(x)$ the slope of the graph must be $f(x, y)$. As we've already discussed a little bit, this gives an alternate way of understanding the graph of the solution $y(x)$ even without ever actually finding a formula for $y(x)$.

Consider a **slope field** near the point (x_0, y_0): at each nearby point (x, y), assign the slope given by $f(x, y)$. You can represent this in a picture by using small line segments placed at representative points (x, y), having slopes $m = f(x, y)$. (An analogous concept is a **vector field**, which assigns vectors to points. You've possibly discussed that notion in the context of electric, magnetic, or gravitational force fields, for physics and math applications. In the present case we're using fields of slopes rather than vectors.)

Exercise 1: Consider the DE $\dfrac{dy}{dx} = x - 3$, and then the IVP with $y(1) = 2$.

a) Fill in (by hand) segments with representative slopes, to get a picture of the slope field for this DE, in the rectangle $0 \le x \le 5, 0 \le y \le 6$ below. Notice that in this example the value of the slope field only depends on x, so that all the slopes will be the same on any vertical line (having the same x-coordinate). (In general, curves on which the slope field is constant are called **isoclines**, since "iso" means "the same" and "cline" means inclination.) Since the slopes are all zero on the vertical line for which $x = 3$, I've drawn a bunch of horizontal segments on that line in order to get started.

b) Use the slope field to create a qualitatively accurate sketch for the graph of the solution to the IVP above, without ever resorting to a formula for the solution function $y(x)$.

c) This is a DE and IVP we can solve via antidifferentiation. Find the formula for $y(x)$ and compare its graph to your sketch in (b).
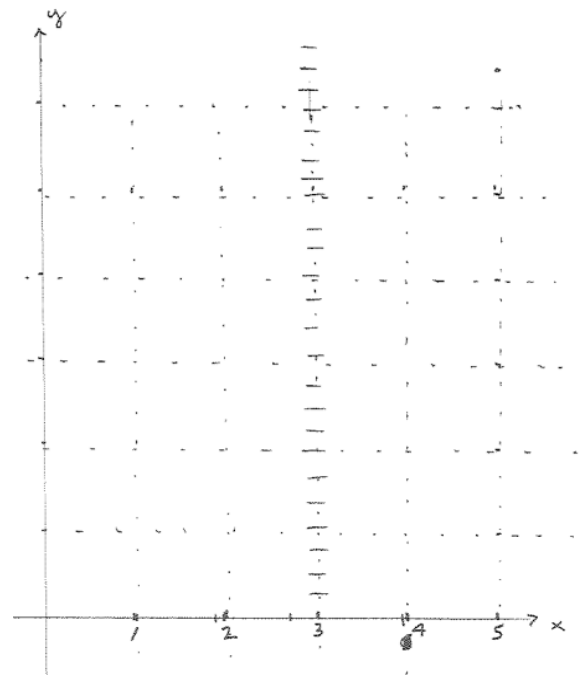
The procedure of drawing the slope field $f(x, y)$ associated to the differential equation $y'(x) = f(x, y)$ can be automated. And, by treating the slope field as essentially constant on small scales, i.e. using

$$\frac{\triangle y}{\triangle x} \approx \frac{dy}{dx} = f(x, y)$$

one can make discrete steps in x and y, starting from the initial point (x_0, y_0), by picking a step size $\triangle x$ and then incrementing y by

$$\triangle y = f(x, y) \triangle x.$$

In this way one can *approximate* solution functions to initial value problems, and their graphs. You can find an applet to do this by googling "dfield" (stands for "direction field", which is a synonym for slope field).

Here's a picture like the one we sketched by hand on the previous page. The solution graph was approximated using numerical ideas as above (but more sophisticated), and this numerical technique works for much more complicated differential equations, e.g. when solutions exist but don't have closed form formulas. The program "dfield" was originally written for Matlab, and can also be downloaded to run inside that package.
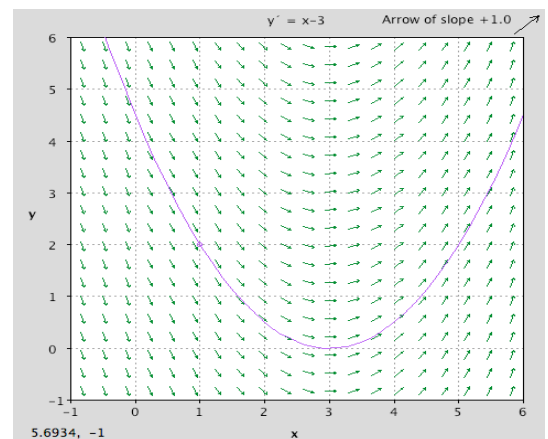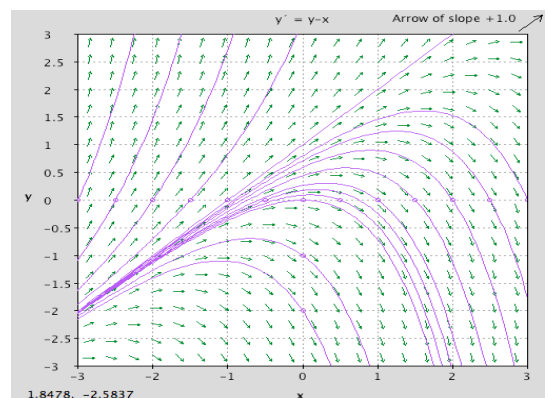
Exercise 2: Consider the IVP

$$\frac{dy}{dx} = y - x$$
$$y(0) = 0$$

a) Check that $y(x) = x + 1 + C e^x$ gives a family of solutions to the DE (C=const). Notice that we haven't yet discussed a method to derive these solutions, but we can certainly check whether they work or not.

b) Solve the IVP by choosing appropriate C.

c) Sketch the solution by hand, for the rectangle $-3 \le x \le 3, -3 \le y \le 3$. Also sketch typical solutions for several different C-values. Notice that this also gives you an idea of what the slope field looks like. How would you attempt to sketch the slope field by hand, if you didn't know the general solutions to the DE? What are the isoclines in this case?

d) Compare your work in (c) with the picture created by dfield at the bottom of this page.

e) Recreate the automated picture with the dfield applet, located at

http://math.rice.edu/~dfield/dfpp.html

If you have a copy of Matlab, you can also download and use a matlab version of this program.

<u>Exercise 3</u>: Consider the differential equation

$$\frac{dy}{dx} = 1 + y^2 \ .$$

a) Use separation of variables to find solutions to this DE...the separation of variables algorithm that we justified in Wednesday's notes, as a way to "do the chain rule backwards".

b) Use the slope field below to sketch some solution graphs. Are your graphs consistent with the formulas from a? (You can sketch by hand, I'll use "dfield" on my browser.)

c) Explain why each IVP has a solution, but this solution does not exist for all x.
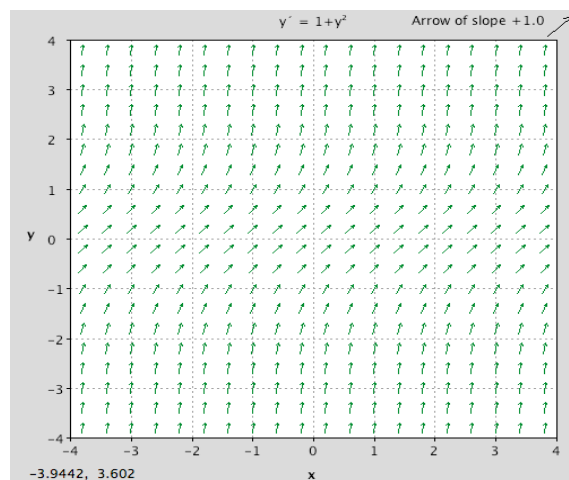
Exercise 4) Use separation of variables to solve the IVP

$$\frac{dy}{dx} = y^{\left(\frac{2}{3}\right)}$$

$$y(0) = 0$$

4b) But there are actually a lot more solutions to this IVP! (Solutions which don't arise from the separation of variables algorithm are called <u>singular</u> solutions.) Once we find these solutions, we can figure out why separation of variables missed them.

4c) Sketch some of these singular solutions onto the slope field below.
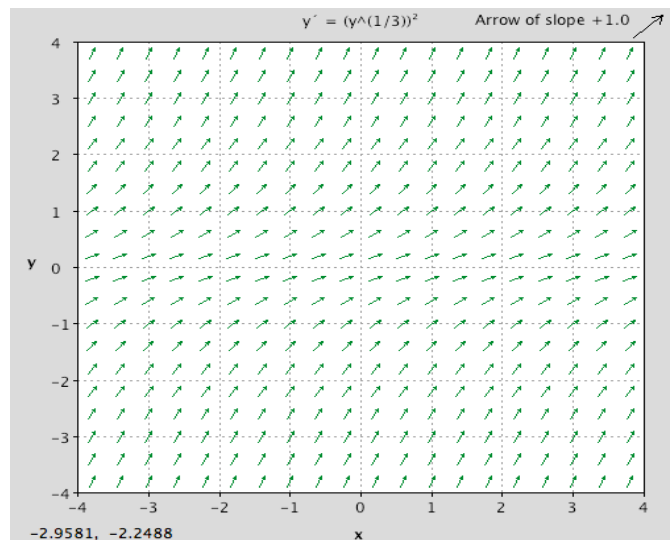
Here's what's going on (stated in section 1.3 of text and partly proven in Appendix A.)

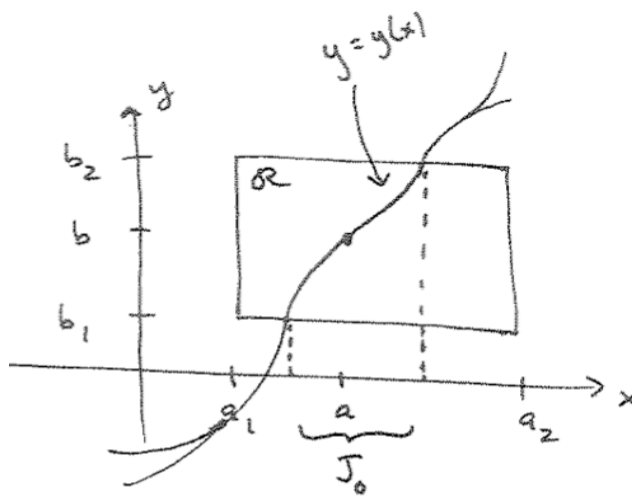<u>Existence - uniqueness theorem for the initial value problem</u>

Consider the IVP

$$\frac{dy}{dx} = f(x, y)$$
$$y(a) = b$$

• Let the point (a, b) be interior to a coordinate rectangle $\mathcal{R} : a_1 \leq x \leq a_2, b_1 \leq y \leq b_2$ in the x-y plane.

• <u>Existence:</u> If $f(x, y)$ is <u>continuous</u> in $\mathcal{R}$ (i.e. if two points in $\mathcal{R}$ are close enough, then the values of f at those two points are as close as we want). Then there exists a solution to the IVP, defined on some subinterval $J \subseteq [a_1, a_2]$.

• <u>Uniqueness:</u> If the partial derivative function $\dfrac{\partial}{\partial y} f(x, y)$ is also continuous in $\mathcal{R}$, then for any subinterval $a \in J_0 \subseteq J$ of x values for which the graph $y = y(x)$ lies in the rectangle, the solution is unique!

See figure below. The intuition for existence is that if the slope field $f(x, y)$ is continuous, one can follow it from the initial point to reconstruct the graph. The condition on the y-partial derivative of $f(x, y)$ turns out to prevent multiple graphs from being able to peel off.

Exercise 5: Discuss how the existence-uniqueness theorem is consistent with our work in all of today's exercises, where we were able to find explicit solution formulas because the differential equations were actually separable.